



Multiparticle Systems. The Algebra of Integrals and Integrable Cases

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Abstract—Systems of material points interacting both with one another and with an external field are considered in Euclidean space. For the case of arbitrary binary interaction depending solely on the mutual distance between the bodies, new integrals are found, which form a *Galilean momentum* vector. A corresponding algebra of integrals constituted by the integrals of momentum, angular momentum, and Galilean momentum is presented. Particle systems with a particle-interaction potential homogeneous of degree $\alpha = -2$ are considered. The most general form of the additional integral of motion, which we term the Jacobi integral, is presented for such systems. A new nonlinear algebra of integrals including the Jacobi integral is found. A systematic description is given to a new reduction procedure and possibilities of applying it to dynamics with the aim of lowering the order of Hamiltonian systems. Some new integrable and superintegrable systems generalizing the classical ones are also described. Certain generalizations of the Lagrangian identity for systems with a particle-interaction potential homogeneous of degree $\alpha = -2$ are presented. In addition, computational experiments are used to prove the nonintegrability of the Jacobi problem on a plane.

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INTRODUCTION

We consider here systems of material points interacting both with one another and with an external field, in Euclidean space. In particular, many problems of celestial mechanics are reduced to such problems. For the case of arbitrary pair interaction of bodies that depends solely on the distance between them, we find new integrals that form a *Galilean-momentum* vector. A corresponding algebra of integrals constituted by the integrals of momentum, angular momentum, and Galilean momentum is presented. To determine the possibility of using these integrals for a reduction, we follow the classical procedure based on the general Lie–Cartan theorem and related to the Lie algebra of the first integrals. In turn, this procedure is in fact based on the Routh reduction procedure. Note that the above-noted algebra has not been considered in celestial mechanics, since all the previously known problems had, as a rule, a Galilean invariance and could be solved by reducing the problem to the center-of-mass system. There are, however, some problems concerning the dynamics of particles in constant-curvature spaces without Galilean invariance and with the components of the Galilean momentum becoming adiabatic invariants. This makes it possible to apply the Hamiltonian perturbation theory to problems of celestial mechanics in constant-curvature spaces.

The systems of particles whose interaction is specified by a potential homogeneous of degree $\alpha = -2$ prove to have an additional symmetry. Such systems were already considered by Newton and (in a more systematic form) by Jacobi, Chazy, Woronetz. An additional, hidden symmetry exists for such systems, and a first integral of motion, which we will term the Jacobi integral, corresponds to this symmetry. This integral was noted repeatedly, starting from Jacobi’s study; we present it, however, in a more general form. In addition, we present a new nonlinear algebra of integrals that includes the Jacobi integral. After finding the algebra of integrals, we systematically describe the new reduction procedure and possibilities of applying it to dynamics with the aim of reducing the orders of Hamiltonian systems. This procedure is fairly general in its form and is related precisely to the degree of homogeneity of the potential, $\alpha = -2$. Unfortunately, ways of applying this procedure has not yet been found in a general form. We only note that this reduction differs substantially from the Routh reduction, being related to the integral of motion quadratic with respect to momentum and being non-Hamiltonian in our representation.

We also present here certain new integrable and superintegrable systems generalizing some classical ones. A number of generalizations of the Lagrange identity is given for systems with potentials homogeneous of degree $\alpha = -2$. Numerical experiments are also carried out to prove the nonintegrability of the Jacobi problem on a plane. In addition, some new unresolved issues and avenues of research in celestial mechanics are noted in our article.

1. N-BODY PROBLEM

1.1. Integrals of Motion, Reduction, and the Algebra of Integrals

Consider the classical N -body problem with arbitrary masses m_i , $i = 1, \dots, N$, moving in the space \mathbb{R}^n under the action of potential forces depending on the mutual distances between the bodies.

We denote the position and momentum of the n th body in a standard way, as n -dimensional vectors \mathbf{r}_i and \mathbf{p}_i . Then the equations of motion in a Hamiltonian form are

$$\begin{aligned} \dot{\mathbf{r}}_i &= \frac{\partial H}{\partial \mathbf{p}_i}, & \dot{\mathbf{p}}_i &= -\frac{\partial H}{\partial \mathbf{r}_i}, & i &= 1 \dots N, \\ H &= \frac{1}{2} \sum \frac{\mathbf{p}_i^2}{m_i} + U(|\mathbf{r}_i - \mathbf{r}_j|), \end{aligned} \quad (1.1)$$

where the Hamiltonian H specifies the total energy of the system.

It is well known that Eqns (1) are invariant with respect to the group of motions $E(n)$ of the space \mathbb{R}^n . Therefore, according to the Noether theorem [24], they admit $n + \frac{n(n-1)}{2}$ first integrals of motion

$$\mathbf{P} = \sum \mathbf{p}_i, \quad M_{\mu\nu} = \sum_i r_\mu^{(i)} p_\nu^{(i)} - r_\nu^{(i)} p_\mu^{(i)}, \quad (1.2)$$

which express the conservation of the total momentum and total angular momentum of the system of bodies. From here on, we number the bodies with Latin subscripts and the components of vectors and tensors with Greek subscripts.

In addition, Eqs. (1.1) are invariant with respect to transformations to a coordinate system uniformly moving relative to the original one. Together with the motions of the space, these transformations form a Galilean-transformation group. The corresponding (according to the Noether theorem) integrals of motion

$$\mathbf{c} = \mathbf{R} - \frac{\mathbf{P}}{\sum m_i} t, \quad \text{where} \quad \mathbf{R} = \frac{\sum m_i \mathbf{r}_i}{\sum m_i} \quad (1.3)$$

are not autonomous (explicitly time-dependent) and are related to uniform, rectilinear motion of the center of mass \mathbf{R} .

Integrals (1.2) and (1.3) admit a reduction by $n + \frac{1}{2} \left(\frac{n(n-1)}{2} + \left[\frac{n}{2} \right] \right)$ degrees of freedom (from here on, we denote the integer part of a number with square brackets).

The classical reduction procedure in the case under consideration consists of two stages:

1. A reduction over the group of translations and Galilean group by n degrees of freedom. As suggested by Jacobi [1], it can be carried out using a transition to barycentric (fixed to the center of mass) coordinates

$$\xi_k = \mathbf{r}_{k+1} - \mathbf{R}_k, \quad k = 1, \dots, N-1, \quad \text{where} \quad \mathbf{R}_k = \sum_{i=1}^k m_i \mathbf{r}_i / \sum_{i=1}^k m_i. \quad (1.4)$$

2. A reduction over the group of rotations, $SO(n)$, by $\frac{1}{2} \left(\frac{n(n-1)}{2} + \left[\frac{n}{2} \right] \right)$ degrees of freedom. In the problem of motion of three bodies in \mathbb{R}^3 , this procedure is called the *elimination of a node* and can be done in various ways [2–4].

Note that the integrals (1.3) are not autonomous. In particular, this prevents one to apply the algebraic method of reduction of the order found by Poincaré (a Hamiltonian version of Routh's order-reduction procedure) and generalized by E. Cartan for the case of a noncommutative algebra of integrals [5].

We give here the principal theorem on which the algebraic reduction procedure rests. Consider a Hamiltonian system with first (autonomous) integrals F_1, \dots, F_k such that $\{F_i, F_j\} = a_{ij}(F_1, \dots, F_k)$. The set of integrals F_1, \dots, F_k specifies a natural mapping $F : M \rightarrow \mathbb{R}^k$, and the functions a_{ij} are generally nonlinear. As Cartan has shown, the following theorem is valid in this case.

Theorem 1 (Lie–Cartan). *Let a point $c \in \mathbb{R}^k$ be not the critical value of the mapping F , and the rank of the matrix $\|a_{ij}\|$ is constant in its neighborhood. Then, in a small neighborhood $U \subset \mathbb{R}^k$ of the point c one can find k independent functions $\varphi_s : U \rightarrow \mathbb{R}$ such that they satisfy the relationships*

$$\{\Phi_1, \Phi_2\} = \dots = \{\Phi_{2q-1}, \Phi_{2q}\} = 1, \quad (1.5)$$

whereas $\{\Phi_i, \Phi_j\} = 0$ for all the remaining brackets. The quantity $2q$ is equal to the rank of the matrix $\|a_{ij}\|$.

The proof of this theorem can be found in [5].

In general, the Lie–Cartan theorem extends some observations associated with the order reduction in celestial mechanics, which trace back to Lagrange, Liouville, Bertrand, Bour, Lie, and many others. We shall not consider here this multilateral problem; however from this point of view one can note that the reduction procedure itself and the sense of finding the algebra of integrals are the following. If one finds the first integrals of the system and describes their algebra then he can calculate by how many degrees of freedom the order of the system can be decreased. Also the constructive reduction of the system can be done.

It should be noted that in the classical three-body problem all these reasons were used already by Lagrange who introduced mutual distances between bodies and corresponding velocity projections for the reduction of this problem and also by Bour in the most systematic and detailed form in his remarkable paper [2], where he essentially used commutator tables and the Poisson structure. In a sense the Bour paper was poorly evaluated from the point of view of the Lie procedure; although in this paper he anticipated many aspects, developed later in [6] and [7]. Most symmetric form of the classical procedure of reduction is given in [8, 9].

What is a sense of this procedure? If one has the algebra of integrals then in the following he should find functions, invariant w.r.t. the action of this algebra of integrals. These functions are coordinates of the reduced system.

If the algebra of integrals is linear, i.e., it is a Lie algebra, then the coordinates are simply invariants of an action of the corresponding Lie group. If the algebra of integrals is nonlinear then one should find invariant functions of phase flows, generated by first integrals. At first one specifies a natural set of functions then enriched by the so called Jacobi method. New functions are generated by commuting of known functions or their combinations. This process occurs until the set of functions is not closed w.r.t. commutation; thus one gets the closed algebra. As a rule the reduced system is defined on this algebra. It is necessary however that the Hamiltonian can be fully expressed through these functions, which are variables of the reduced system. This algorithm is fully described, for example, in the book [10], where there is also an application of this method to the vortex theory.

The constructed algebra of the reduced system can degenerate, i.e., it can contain Casimir functions. In principle, such system is already reduced and can be investigated. For instance, for some problems in the perturbation theory, it is more comfortable to have a canonical representation of the reduced system. In this case a reduction onto the symplectic leaf is fulfilled. For Lie algebras this technique is smooth-running due to Bour. For a nonlinear algebra it is rather an art and one needs some additional reasons, which are developed below for systems with a homogeneous potential of the degree $\alpha = -2$. Here the standard symplectization considerations do not work, but there is another construction, connected with a projection onto a sphere and changing of time. Note that we try to perform all considered procedures in an explicit form.

As an example consider the three-body motion in the space \mathbb{R}^3 . In this case autonomous integrals (1.2) form the algebra $e(3)$ and in theorem 1 one should put $k = 6$, $q = 2$. The Lie–Cartan theorem makes possible the reduction by four degrees of freedom, while the classical reduction procedure allows (see page 20) to eliminate five degrees of freedom. Thus for the correct Lie–Cartan reduction one needs another independent autonomous integral that is in involution with (1.2).

For an arbitrary N -body system in the space \mathbb{R}^n there is one another set of autonomous first integrals of motion; for the three-body problem in \mathbb{R}^3 it was found by S. Lie in [11]. In the same paper he wrote commutator relations for these integrals and constructed the set of five independent integrals in involution. Thus he pointed out the possibility to reduce five degrees of freedom in the three-body problem in \mathbb{R}^3 by algebraic methods.

In the following he planned to transpose algebraic methods of analysis of free motion and reduction onto celestial mechanics in constant curvature spaces. Particularly, in a footnote in his paper [11] he wrote: “*In the following I shall construct mechanics on n -dimensional manifolds of constant curvature. Integrals generated by a free motion of the corresponding space can be derived according to general principles, which I shall describe later. The present paper shows the best way of using these integrals. I do not know if the corresponding theory exists.*”. Due to some reasons he reject later this thought (a reader can learn the scientific biography of S. Lie from [12, 13]).

The following theorem generalizes the S. Lie results.

Theorem 2. *Besides of (1.2) equations (1.1) of the N -body motion in \mathbb{R}^n admit also $\frac{n(n-1)}{2}$ autonomous integrals of motion*

$$Q_{\mu\nu} = R_\mu P_\nu - R_\nu P_\mu, \quad \mu, \nu = 1 \dots n. \quad (1.6)$$

Furthermore these integrals are connected with each other and integrals \mathbf{P} by relations

$$\sum_{\langle \mu, \nu, \rho \rangle} \varepsilon_{\mu\nu\rho} Q_{\mu\nu} P_\rho = 0, \quad (1.7)$$

where $\mu \neq \nu \neq \rho \neq \mu$ and summation corresponds to the cyclic transpositions of indices.

Proof. We prove the first part of the theorem by the elimination of time from different pairs of integrals (1.3). Actually

$$Q_{\mu\nu} = c_\mu P_\nu - c_\nu P_\mu = R_\mu P_\nu - R_\nu P_\mu \quad (1.8)$$

and constraint equations (1.7) follows directly from the form of integrals (1.6). \square

Note that integrals (1.6) constitute the tensor of the *average* angular momentum of the system, i.e., they correspond to the angular momentum of the body with the mass $\mu = \sum m_i$ in the center of mass of the system moving with the momentum equals the full momentum of the system. The existence of tensor integral (1.6) (as well as tensor integral (1.3)) is connected with the invariance of equations of motion w.r.t. the Galilean group, therefore we shall call it the *Galilean momentum*. Note also, that these integrals depend upon the choice of the coordinate system. Thus the classical transition to the barycentric coordinates can be seen as a constructive form of the reduction w.r.t. integrals \mathbf{P} and \mathbf{Q} .

Specify the Poisson brackets of integrals (1.2) and (1.6). From here on we shall restrict ourselves with the N -body motion in \mathbb{R}^3 . Generalization onto higher dimensions is not difficult and can be done according, for example, [14]. We shall pass from tensors to vectors of the full and the Galilean momentum $M_{\mu\nu} = \varepsilon_{\mu\nu\rho} M_\rho$, $Q_{\mu\nu} = \varepsilon_{\mu\nu\rho} Q_\rho$ in the standard way. The Poisson brackets now have the form

$$\begin{aligned} \{M_\mu, M_\nu\} &= \varepsilon_{\mu\nu\rho} M_\rho, & \{M_\mu, Q_\nu\} &= \varepsilon_{\mu\nu\rho} Q_\rho, & \{M_\mu, P_\nu\} &= \varepsilon_{\mu\nu\rho} P_\rho, \\ \{Q_\mu, Q_\nu\} &= \varepsilon_{\mu\nu\rho} Q_\rho, & \{Q_\mu, P_\nu\} &= \varepsilon_{\mu\nu\rho} P_\rho, & \{P_\mu, P_\nu\} &= 0. \end{aligned} \quad (1.9)$$

The algebra of integrals (1.9) has three Casimir functions

$$K_0 = (\mathbf{Q}, \mathbf{P}), \quad K_1 = (\mathbf{M}, \mathbf{P}), \quad K_2 = |\mathbf{M} - \mathbf{Q}|^2, \quad (1.10)$$

one of them necessarily equals null $K_0 = 0$ that coincides with constraint equation (1.7).

Consider the decreasing of the order of the system (1.1) using integrals $\mathbf{M}, \mathbf{P}, \mathbf{Q}$. In the case \mathbb{R}^3 the following theorem is valid.

Theorem 3. *Integrals (1.2) and (1.6) allow to reduce five degrees of freedom in the N -body problem in \mathbb{R}^3 .*

Proof. Let us calculate the rank of Jacobian $\text{rank}\left(\frac{\partial(\mathbf{M}, \mathbf{P}, \mathbf{Q})}{\partial(\mathbf{r}_i, \mathbf{p}_i)}\right) = 8$ and the following $\text{rank}(\{(\mathbf{M}, \mathbf{P}, \mathbf{Q}), (\mathbf{M}, \mathbf{P}, \mathbf{Q})\}) = 6$ of the matrix of brackets of integrals. From these ranks one can conclude that there are eight independent integrals in the set $\mathbf{M}, \mathbf{P}, \mathbf{Q}$ of nine integrals and the algebra of integrals (1.9) contains two nontrivial Casimir functions. Therefore in the general case one can construct $2 + \frac{6}{2} = 5$

independent integrals in involution using integrals $\mathbf{M}, \mathbf{P}, \mathbf{Q}$. According to S. Lie, one can choose the following integrals P_1, P_2, P_3, K_1 and K_2 . Thus using the Liouville theorem one can reduce five degrees of freedom in the N -body problem on a sphere that correspond to the classical reduction procedure on the base of barycentric coordinates. \square

1.2. The Two-Body Problem in Constant Curvature Spaces. Formulation of the Problem

As it was said already above, the S. Lie paper [11] (dealing with the plane N -body problem) contains a footnote, which mentions the N -body problem in the constant curvature space. It is well known that in this case there are no analogs of the Galilean group and the center of mass integral. However the center of mass notion in a curved space can be postulated (generally in different ways); this was done, for example, by Zhukovski in [15] and by G.A. Galperin in [16].

The fundamental paper of Killing [17] contains main results, problems formulations and systematic exploration of mechanics of points and rigid bodies in curved spaces. Many results of this paper were later rediscovered; the survey of modern researches in this direction can be found, for instance, in collection of papers [18].

One should note that explorations in mechanics in curved spaces were quite popular at the end of the 19th and the beginning of the 20th centuries. It was connected with creation of the noneuclidean geometry. Already its creators Lobachevski, Riemann and Beltrami formulated the problem of describing not only geometry and kinematics, but also the dynamic in curved spaces. The generalization of the Newton attraction law for spaces of constant curvature appeared in this way. At the beginning of the 20th century the interest to this topic declined due to the creation of special and later general relativity. In general relativity the notion of a curvature plays an essential role — the Riemann curvature tensor appears in equations of general relativity. However the sense of a curvature in the general relativity differs from the considering situation; it plays there a dynamic role, connected with gravitational mass.

Already in Einstein times many mathematicians, particularly Shazy and Levy–Civita, tried to construct a relativistic celestial mechanics and formulated the N -body problem in the relativistic space. The space itself remained Euclidean one, and the presence of a curvature, generated by the gravitation, led to some corrections in the gravitational law. In particular, the inverse square potential appeared in such a way. It will be studied below in details. Newton was the first to study potentials with correcting terms, namely, he pursued the question of the form of the central field potential with the property that all the orbits are closed in a uniformly rotating frame of reference. Correcting terms may also be added to account for planet's nonsphericity.

Besides of the classical relativistic approach, when the Euclidean space becomes curved in the presence of gravitational masses, one can consider the motion in the initially curved space. In this case we can also found equations of the gravitational and relativity theory, but in a curved space. For simplicity, one should consider spaces of constant curvature. Already Einstein, Eddington and de Sitter proposed the corresponding static models of a real world. In particular, the spherical space \mathbb{S}^3 is called the “Einstein world”.

The introduction of a curvature into the gravitational theory seems appropriate. Indeed, one of the most significant achievement of the gravitation theory, is the Friedman idea on isotropic universes. Within the gravitational theory a curvature can be considered as a phone distribution of the mass of a universe. The modern comprehensive analysis of gravitation theories in curved spaces can be found in the N.A. Chernikov paper [19], where he considered the Lobachevski space instead of the Euclidean one. In his paper N.A. Chernikov generalized gravitational equations and investigated the simplest problems in the Lobachevski space. In the same paper he gave a new reasoning for introducing a phone curvature, connected with the tensor analysis and the independence of the pseudotensor of a gravitational energy upon the choice of a coordinate system.

As in the Euclidean space, using the limiting procedure in equations of the general relativity for a curved space, one can get equations of Newtonian mechanics in a constant curvature space. In other words, the classical mechanics in a constant curvature space is intermediate between Newtonian mechanics and general relativity in curved spaces. From one hand it is much simpler than relativistic mechanics and from another hand it can explain some dynamical relativistic effects, for example, the displacement of the perihelion of the Mercury orbit. Note, that the Einstein explanation of the displacement of the perihelion of the Mercury orbit finally is reduced to a perturbation of the interaction potential of the plane problem, connected with a space curvature. In this sense the

explanation from the point of view of the classical mechanics in a constant curvature space can exist and is more rigorous in some sense. This and some other problems of the celestial mechanics in constant curvature spaces were considered in our papers [20–23].

Consider the N -body problem in constant curvature spaces. To do so we shall parametrize the sphere \mathbb{S}^3 (the pseudosphere \mathbb{L}^3), using the redundant coordinates of the four dimensional Euclidean space \mathbb{R}^4 (the Minkowski space \mathbb{M}^4) with the following constraint

$$\Phi(q) = \frac{1}{2} (g_{\mu\nu} q^\mu q^\nu \mp R^2) = \frac{1}{2} (\langle \mathbf{q}, \mathbf{q} \rangle \mp R^2) = 0, \quad (1.11)$$

where $g = \text{diag}(1, 1, 1, 1)$ ($g = \text{diag}(-1, 1, 1, 1)$) is the corresponding metric. The upper sign here and below corresponds to the sphere and the lower sign to a pseudosphere. The metric of the corresponding space $\langle \cdot, \cdot \rangle$ induces the metric of the sphere \mathbb{S}^3 and the Lobachevski metric on the pseudosphere \mathbb{L}^3 .

The motion of N particles in the redundant coordinates in the potential $U(\mathbf{q}_1, \dots, \mathbf{q}_N)$ is described by the Lagrangian

$$\mathcal{L} = \frac{1}{2} \sum_{i=1}^N \langle \dot{\mathbf{q}}_i, \dot{\mathbf{q}}_i \rangle - U(\mathbf{q}_1, \dots, \mathbf{q}_N)$$

with the constraint (1.11). Using the Hamiltonian formalism of constraint systems [24], we get the Hamiltonian

$$\mathcal{H} = \frac{1}{2} \sum_{i=1}^N \left(\langle \mathbf{p}_i, \mathbf{p}_i \rangle - \frac{\langle \mathbf{p}_i, \mathbf{q}_i \rangle^2}{\langle \mathbf{q}_i, \mathbf{q}_i \rangle} \right) + U(\mathbf{q}_1, \dots, \mathbf{q}_N). \quad (1.12)$$

The equations of motion $\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}$, $\dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}}$ in variables \mathbf{q} , \mathbf{p} are canonical ones.

It is well known [17, 25–29] that the analog of Newtonian interactive potential for the sphere is

$$U(\mathbf{q}_i, \mathbf{q}_j) = -\gamma \text{ctg } \theta_{ij} = -\gamma \frac{\langle \mathbf{q}_i, \mathbf{q}_j \rangle}{\sqrt{R^2 \mp \langle \mathbf{q}_i, \mathbf{q}_j \rangle^2}}, \quad i, j = 1, \dots, N, \quad i \neq j. \quad (1.13)$$

Recall that the potential (1.13) can be obtain either as the solution of the Laplace–Beltrami equation on the sphere \mathbb{S}^3 , invariant w.r.t. the group $SO(3)$ with a singularity at $\theta = 0$, or from the generalization of the Bertran theorem for the sphere [25, 30]. Thus, the classical N -body problem in constant curvature spaces is described by the following Hamiltonian

$$\mathcal{H} = \frac{1}{2} \sum_{i=1}^N \left(\langle \mathbf{p}_i, \mathbf{p}_i \rangle - \frac{\langle \mathbf{p}_i, \mathbf{q}_i \rangle^2}{\langle \mathbf{q}_i, \mathbf{q}_i \rangle} \right) - \gamma \sum_{i>j=1}^N \frac{\langle \mathbf{q}_i, \mathbf{q}_j \rangle}{\sqrt{R^2 \mp \langle \mathbf{q}_i, \mathbf{q}_j \rangle^2}}. \quad (1.14)$$

The two-body problem in constant curvature spaces is nonintegrable [22]. In the limiting case $R \rightarrow \infty$ it transforms into the integrable classical two-body problem. Note that under the inverse transition from the plane to the curved problem integrals of motion are conserved (for example the angular momentum), but the Galilean invariance disappears. In this connection it would be interesting to construct the standard perturbation theory for the plane problem using the space curvature as a parameter and integrals of the Galilean momentum (or their combinations) as adiabatic invariants.

2. NATURAL SYSTEMS WITH A HOMOGENEOUS POTENTIAL OF DEGREE $\alpha = -2$

2.1. Integrals of Motion

Consider a natural system with the potential $U_\alpha(\mathbf{r})$, which is the homogeneous function of degree α w.r.t. variables r_i , $i = 1, \dots, N$

$$H = \frac{1}{2} \sum_{i=1}^N \frac{p_i^2}{m_i} + U_\alpha(\mathbf{r}). \quad (2.1)$$

It is well known that the Euler identity

$$\left(\mathbf{r}, \frac{\partial U_\alpha}{\partial \mathbf{r}}\right) = \alpha U_\alpha. \tag{2.2}$$

is valid for the potential $U_\alpha(\mathbf{r})$.

The *Lagrange formula*

$$\ddot{I} = 4H - 2(\alpha + 2)U_\alpha \tag{2.3}$$

describes the evolution of the central moment of inertia $I = \sum_{i=1}^N m_i r_i^2$ for these systems. Some generalizations of the Lagrange formula for constraint systems with potential energy quasihomogeneous in coordinates and for continuum interacting particles can be found in the recent paper [31].

In the case $\alpha = -2$ the Lagrange formula is simplified and the equation for the moment of inertia can be solved in the explicit form

$$I(t) = 2Ht^2 + at + b, \tag{2.4}$$

where a and b are constants of integration. Equation (2.4) leads to the following proposition for the N -body system with the potential $U_{-2}(\mathbf{r})$.

Proposition 1 (Jacobi). *For a negative energy $H < 0$ all particles will collide in a finite time; for a positive energy $H > 0$ at least one mutual distance will increase indefinitely at $t \rightarrow \pm\infty$.*

Constants of integration a and b are nonautonomous (time dependent) integrals of motion. For the first time their existence was mentioned by Jacobi [32], who studied the three-body problem on a line with the potential $U = \sum_{i,j=1,i>j}^N \frac{m_i m_j}{(x_i - x_j)^2}$. For an arbitrary potential $U = U_{-2}(\mathbf{r})$ the integrals a and b can be written in the explicit form

$$a = 2(\mathbf{r}, \mathbf{p}) - 4Ht, \quad b = 2Ht^2 - 2(\mathbf{r}, \mathbf{p})t + I, \tag{2.5}$$

where H is the Hamiltonian of the system.

Note that on a fixed level of the energy and for $\alpha = -2$ the Lagrange formula (2.3) has the form of the Newton law with a constant force. This equation admits the evident integral of energy

$$h = \frac{1}{2}\dot{I}^2 - 4IH = \frac{a^2}{2} - 4bH, \tag{2.6}$$

which is also an integral for the whole system. This leads to the following theorem

Theorem 4. *The natural system (2.1) with the potential $U_{-2}(\mathbf{r})$ admits the first integral of motion*

$$J = -h/2 = 2IH - (\mathbf{r}, \mathbf{p})^2, \tag{2.7}$$

which can be treated as the energy of the radial motion of the system under consideration.

One can show that equations of motion of the system (2.1) with the potential $U_{-2}(\mathbf{r})$ are invariant w.r.t. the following one-parametric group

$$G_a : \begin{cases} \mathbf{r} \rightarrow e^\lambda \mathbf{r}, \\ t \rightarrow e^{2\lambda} t; \end{cases} \quad G_b : \begin{cases} \mathbf{r} \rightarrow \frac{\mathbf{r}}{1 - \lambda t}, \\ t \rightarrow \frac{t}{1 - \lambda t}, \end{cases} \tag{2.8}$$

transforming coordinates and time, where λ is the parameter of dilatation. According to the Noether theorem these transformations generate the nonautonomous integrals a and b (2.5). The Poisson brackets of these integrals with each other and the Hamiltonian have the form

$$\{H, a\} = -4H, \quad \{H, b\} = -a, \quad \{a, b\} = -4b; \tag{2.9}$$

they form the Lie–Poisson brackets, corresponding to the algebra $sl(2)$. The nonautonomous integral (2.7) is the Casimir function of this algebra. Though contrary to integrals (2.5) its existence is connected with a hidden symmetry of the problem, not corresponding to any evident group of transformation of the phase space.

Remark 1. Note that $\ddot{I} = 0$ for the moment of inertia I . Thus the following lemma is valid for an arbitrary system of differential equations.

Lemma 1. Let a function $f(\mathbf{r}, \mathbf{p})$ of phase variables \mathbf{r}, \mathbf{p} satisfies the equality $\frac{d^3 f(\mathbf{r}, \mathbf{p})}{dt^3} = 0$. Then, besides of the integral $J_1 = \frac{d^2 f(\mathbf{r}, \mathbf{p})}{dt^2}$, the equations of motion admit the following first integral

$$J_2 / \dot{f} - \dot{f}^2. \quad (2.10)$$

Historical comment 1. For the first time the integral (2.7) was found by Jacobi [33] while studying the motion of a particle in \mathbb{R}^3 under an action of central forces with an addition of an arbitrary homogeneous potential of the degree $\alpha = -2$, i.e.,

$$H = \frac{1}{2} \mathbf{p}^2 + V(|\mathbf{r}|) + U_{-2}(\mathbf{r}), \quad \mathbf{p}, \mathbf{r} \in \mathbb{R}^3. \quad (2.11)$$

In the followings using the Lax representation some authors [34–38] showed the integrability of the Calogero–Moser system and some its generalizations. These systems also enter into the subject of the present paper. However the integral (2.7) was not explicitly mentioned though it is contained in the Lax representation.

For the first time the integral (2.7) was written by Albouy and Chenciner in [6] in the most general form for an arbitrary homogeneous potential of the degree $\alpha = -2$.

Historical comment 2. Homogeneous potentials of degree $\alpha = -2$ were considered by Newton searching a law of motion, corresponding to closed orbits in rotating coordinate systems. It happens that this case corresponds to the sum of the classical potential with a term, which is an inverse square in the distance between bodies. The contemporary of Newton and the publisher of the second edition of the “Principia” R. Cotes considered this potential separately and showed that motions in this potential are spirals, later called the Cotes spirals. Generally, additions to the interaction potential of the form $U(\mathbf{r}) \sim |\mathbf{r}|^{-2}$ were considered permanently in celestial mechanics as corrections terms, which are necessary for explaining some facts, lying beyond the Newton theory. In particular these additions were considered already by A. Clairaut for explaining the motion of the apse of the lunar orbit. In the followings, P. Laplace developed methods of celestial mechanics and proved that the law of universal gravitation fully explains the planets motion if we represent their mutual perturbations by mathematical series.

For a discussion of Einstein’s corrections to the Newtonian potential (introduced as an attempt to explain the advance of the perihelion of Mercury’s orbit) see the survey paper [39]. Besides, additions of the same kind are considered in the relativistic two-body problem (the so called Manev problem). Its nonintegrability was shown in [40]. The quantum mechanical problem with a similar potential was considered in the G. Shortley paper [41].

One of the most systematic analysis of the N –body problem with the potential $\alpha = -2$ is due to P.V. Woronetz [4, 42–44]. Specifically, he considered in details equilateral configurations in the three-body problem and found some interesting particular solutions of the N –body problem, which can be thoroughly analyzed by constructing quadratures [42]. In [4, 43], for both spatial and plane N –body problem he obtained new non-trivial solutions (missing in the case of the Newtonian potential). In [44] Woronetz examines the case of $\alpha = -2$ with some additional constants imposed on the system. A further analysis of these solutions is contained in Banachiewicz [45], Bilimowitch [46], Longley [47].

Later P.V. Woronetz ideas were developed by a relatively unknown (aborad) Ukrainian scientist Yu.D. Sokolov [48–50], who pointed out some new particular solutions of the N –body problem and some cases of its integrability. For instance, he found the case of integrability of the three-body problem on the line, when the interaction between bodies is described by the potential $U(r) = Ar^2 + Br^{-2} + Cr^4$ and showed that for $C = 0$ this problem is integrable in elliptic functions. Sokolov’s results were then further extended by Egervary [51].

The paper by Chazy [52] is an independent intriguing analysis of the N –body problem with $\alpha = -2$. Chazy studies stability of the N –body problem and, in doing so, extends the classical Sundman’s theory to the case of $\alpha = -2$. The paper also contains interesting historical remarks and discussion.

A homogeneous potential of degree $\alpha = -2$ is important for the celestial mechanics and requires a special consideration while studying central and homographic configurations (see for example a recent book by Saari [53]). Besides, already A. Wintner [54] has shown that for this potential the Galilean group is the subgroup of all “inertial” time dependent transformations. He explained by this fact the existence of additional integrals, founded by Jacobi for the potential under consideration.

Note that homogeneous potentials of another form with degree $\alpha = -2$ appear in the problem of a motion of gaseous ellipsoids [55, 56]. It is interesting that in this case similar to the celestial mechanics the reduction procedure, considered below, is also valid. With its help we show the integrability of some problems.

The next theorem gives the most general form of integral (2.7), which generalize both the Jacobi case (2.11) and the Albouy and Chenciner result¹⁾.

Theorem 5. *The natural system (2.1) with the potential*

$$U(\mathbf{r}) = U_{-2}(\mathbf{r}) + V(I) \quad (2.12)$$

admits the first integral of motion

$$J = 2I(H - V) - (\mathbf{r}, \mathbf{p})^2. \quad (2.13)$$

Proof. The second time derivative of I corresponding to the system (2.1) is

$$\ddot{I} = 4(H - V) - 4I \frac{\partial V}{\partial I}. \quad (2.14)$$

On a fixed level of energy $H = \text{const}$ we again gets the system with one degree of freedom govern by the equation of Newtonian type. Up to a constant integral (2.13) coincides with the energy of system (2.14) and the explicit dependence $I(t)$ is given by

$$t + t_0 = \int \frac{dI}{2\sqrt{-J + 2I(H - V)}}. \quad (2.15)$$

□

2.2. The Reduction Procedure

Note one more time that the classical reduction procedure based in its general form on the Lie–Cartan theorem is closely associated with the presence of integrals, linear in momentum, the Noether theorem and the Lie algebra of first integrals w.r.t. Poisson brackets. This procedure is thoroughly studied (see, for example, [10, 24, 57]). We shall discuss here systematically the new non-Hamiltonian type of reduction, which is essentially different from the classical procedure. The problem of effective reduction in the presence of first integrals quadratic in momentum is very difficult and seems to be unsolvable. But in some particular cases the reduction can be completed.

Thus integral (2.13) allows to reduce one degree of freedom. For the first time it was done by Jacobi while integrating the system (2.11). Later the reduction procedure was discussed by some authors [6, 56] in connection with different concrete dynamical systems. In particular, B. Gaffet made this reduction for the system of three identical bodies on a line with the potential $U = (x_1 x_2 x_3)^{-2/3}$ appeared in a study of the expansion of an ellipsoidal cloud of gas. In these papers as a rule the reduction was done in a nonsymmetric form, essentially connected with local coordinates of the system under consideration.

We shall describe the most symmetric form of this reduction, generalized for an arbitrary system with potential (2.12).

¹⁾This and some other generalizations are presented in Section 6.

Theorem 6. *One degree of freedom can be reduced in the natural system (2.1) with potential (2.12) by the following change of time and coordinates*

$$dt = Id\tau, \quad q_i = \sqrt{\frac{m_i}{I}} r_i, \quad \mathbf{q}, \mathbf{r} \in \mathbb{R}^N. \quad (2.16)$$

The equations of motion in new variables describe the motion of a particle on the $(N-1)$ -dimensional sphere $(\mathbf{q}, \mathbf{q}) = 1$

$$\mathbf{q}'' = -\frac{\partial \tilde{U}_{-2}}{\partial \mathbf{q}} + \lambda \mathbf{q}, \quad (2.17)$$

where $\lambda = (\mathbf{q}, \frac{\partial \tilde{U}_{-2}}{\partial \mathbf{q}}) - \mathbf{q}'^2$ is a Lagrange multiplier,

$$\tilde{U}_{-2}(\mathbf{q}) = IU_{-2}(\mathbf{r}) = U_{-2}\left(\frac{q_i}{\sqrt{m_i}}\right), \quad (2.18)$$

and the prime means a derivative w.r.t. the new time. Integral (2.13) has the form

$$J = \mathbf{q}'^2 + 2\tilde{U}_{-2}(\mathbf{q}) \quad (2.19)$$

and up to a multiplier coincides with the Hamiltonian of the reduced system.

The theorem is proved by a direct substitution of expressions (2.16) into the equations of motion and the integral (2.13).

By the Legendre transformation of the following function on the sphere

$$\mathcal{L} = \frac{1}{2}\mathbf{q}'^2 - \tilde{U}_{-2}(\mathbf{q})$$

one gets a Hamiltonian form of the system (2.17). Up to a multiplier the integral (2.19) coincides with the Hamiltonian of the reduced system.

Note that these reduction procedure is different from the classical Routh's order-reduction and is connected with a hidden symmetry in the extended (including the time) phase space. Indeed, after the substitution (2.16) the integral J becomes a Hamiltonian of a reduced system, when the standard Routh's order-reduction does not change the Hamiltonian, which depends parametrically upon the value of the first integral.

Consider the three-body motion on a line separately. In this case equations of reduced system (2.17) describe the particle motion on the two-dimensional sphere S^2 and using the well-known analogy with the rigid body dynamic [58] one can represent them in a Hamiltonian form with the Poisson brackets, corresponding to the algebra $e(3)$.

In the new variables $\mathbf{M} = \mathbf{q} \times \mathbf{q}'$, $\gamma = \mathbf{q}$ the Hamiltonian has the form

$$H = \frac{1}{2}\mathbf{M}^2 + \tilde{U}_{-2}(\gamma), \quad (2.20)$$

the Poisson brackets

$$\{M_i, M_j\} = e_{ijk}M_k, \quad \{M_i, \gamma_j\} = e_{ijk}\gamma_k, \quad \{\gamma_i, \gamma_j\} = 0 \quad (2.21)$$

degenerate and have two Casimir functions $(\mathbf{M}, \gamma) = c$ and $(\gamma, \gamma) = 1$. Equations (2.20) and (2.21) describe the motion of a spherical top in the potential $\tilde{U}_{-2}(\gamma)$. Besides, it holds $(\mathbf{M}, \gamma) = 0$. This yields the following proposition

Proposition 2. *By the substitution (2.16) the three-body problem on a line with potential (2.12) is reduced to the spherical top problem with the potential $\tilde{U}_{-2}(\gamma) = U_{-2}(\frac{\gamma_i}{\sqrt{m_i}})$ on the zero level of the area integral $c = (M, \gamma) = 0$.*

Note also, that the reduced system on the sphere (2.17) does not contain the summand $V(I)$ from the initial potential (2.12). Thus natural systems with different additions $V(I)$ are reduced to the unique system on the sphere and the following corollary is valid

Corollary 1. *The natural system of the form (2.1) with potential $U(\mathbf{r}) = U_{-2}(\mathbf{r}) + V(I)$ is equivalent to the system without an additional summand $V(I)$ up to an additional quadrature.*

This corollary generalizes the Perelomov result [59] on equivalence of the system of identical particles on a line with the potential $U(\mathbf{r}) = U_{-2}(\mathbf{r}) + \omega^2 \mathbf{r}^2$ and the same system without quadratic summand in the potential. He proved this proposition by the change of coordinates and the time, connected with a rotation system of coordinates.

The following interesting fact will be used later. For the motion of three identical bodies $m_1 = m_2 = m_3 = m$ on a line or equivalently the motion of a one body in \mathbb{R}^3 the transformation (2.16) does not change the vector of kinetic momentum \mathbf{M} that leads to the equality

$$\mathbf{M} = m\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{q} \times \mathbf{q}' \tag{2.22}$$

Thus variables \mathbf{M} are the most convenient for the reduction (2.16)

2.3. The Rosochatius System

As an example, consider the Rosochatius system [60] for the particle motion on the two-dimensional sphere \mathbb{S}^2 with the Hamiltonian

$$H = \frac{1}{2}(M_1^2 + M_2^2 + M_3^2) + \frac{c_1}{\gamma_1^2} + \frac{c_2}{\gamma_2^2} + \frac{c_3}{\gamma_3^2} \tag{2.23}$$

The corresponding equation of motion on the algebra $e(3)$ on the level $(\mathbf{M}, \gamma) = \mathbf{0}$ have three integrals of motion, quadratic in momentum (see, for example, [61])

$$K_i = \frac{1}{2}M_i^2 + \frac{c_j \gamma_k^2}{\gamma_j^2} + \frac{c_k \gamma_j^2}{\gamma_k^2}, \quad \text{where } i = 1 \dots 3. \tag{2.24}$$

The commutation of these integrals on a level $(\mathbf{M}, \gamma) = \mathbf{0}$ leads to another integral

$$F = M_1 M_2 M_3 - \gamma_1 \gamma_2 \gamma_3 \left(\frac{c_1 M_1}{\gamma_1^3} + \frac{c_2 M_2}{\gamma_2^3} + \frac{c_3 M_3}{\gamma_3^3} \right) \tag{2.25}$$

cubic in momentum. It can be shown that in this set there are only three independent integrals. Thus this system is superintegrable that was shown by J. Moser in [62] for a more general case — the Neumann problem with Rosochatius additions.

Consider the change of variables inverse to (2.16). By analogy between the system on the sphere and the N -body problem on a line (see theorem 6) one gets the three body problem on a line with the Hamiltonian

$$H = \frac{1}{2} \left(\frac{p_1^2}{m_1} + \frac{p_2^2}{m_2} + \frac{p_3^2}{m_3} \right) + \frac{c_1^*}{r_1^2} + \frac{c_2^*}{r_2^2} + \frac{c_3^*}{r_3^2}, \tag{2.26}$$

which is separated into the three systems with one degree of freedom.

2.4. The Gaffet System

As another example consider the motion of three identical bodies on a line with the following Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + \frac{c}{(x_1 x_2 x_3)^{2/3}} \tag{2.27}$$

For the first time this system was considered by B. Gaffet studying the particular case of expanding of an ellipsoidal cloud of gas [56]. There he reduced the system under consideration to a particle motion on the two-dimensional sphere and pointed out an additional integral, cubic in momentum. The Lax pair for this problem was found by A.V. Tsiganov in [82]. In the subsequent paper [64] B. Gaffet introduced the rotation of a cloud about one principal axis of inertia into the initial model. For the system on the sphere derived in this way he pointed out the new integral of the six

degree in momentum. Finally, according to the most general result of B. Gaffet from [65, 66] the problem on expanding of an arbitrary vortex-free rotating ellipsoidal cloud of gas is integrable. For this problem he pointed out two additional integrals of the six degree in momentum.

It follows from (2.27) that the potential of the Gaffet system is a homogeneous function of the degree $\alpha = -2$. Therefore this system admits integral (2.7) and using substitution (2.16) can be reduced to a system on the sphere. It is interesting to note that the nonautonomous integral of this problem in the form (10) was independently indicated hundred years later after Jacobi in paper [67].

One can express the Hamiltonian of the reduced system through variables \mathbf{M} , γ in the following form

$$H = \frac{1}{2}(M_1^2 + M_2^2 + M_3^2) + \frac{c}{(\gamma_1\gamma_2\gamma_3)^{2/3}}, \quad (2.28)$$

and the additional integral, cubic in momentum, in the form

$$F_3 = M_1M_2M_3 - 2 \frac{c(\gamma_2\gamma_3M_1 + \gamma_1\gamma_3M_2 + \gamma_1\gamma_2M_3)}{(\gamma_1\gamma_2\gamma_3)^{2/3}}. \quad (2.29)$$

Gaffet wrote integral (2.29) only for system (2.27) on a sphere. Thus for the initial system he showed only the integrability in quadratures. To prove the Liouville integrability of full system (2.27) one should find an additional (besides of (2.7)) third integral of motion that generalize integral (2.29) in initial variables \mathbf{p} , \mathbf{x} . To do so we use the transformation inverse to (2.16) and write integral (2.29) for the full three-body system on a line (2.27)

$$F_3 = \frac{1}{2} \frac{(-p_2x_3 + p_3x_2)(p_2x_1 - p_1x_2)(p_3x_1 - p_1x_3) - c(x_1p_1(x_2^2 - x_3^2) + x_2p_2(x_3^2 - x_1^2) + x_3p_3(x_1^2 - x_2^2))}{(x_1x_2x_3)^{2/3}}. \quad (2.30)$$

It can be easily shown that integrals (2.7) and (2.30) commute with each other. This implies the following theorem

Theorem 7. *System (2.27) admits two additional integrals of motion in involution (2.7) and (2.30) and therefore is Liouville integrable.*

Gaffet constructed the following generalization of system (2.28) with an additional integral of the six degree in momentum. The corresponding Hamiltonian on the sphere has the form

$$H = \frac{1}{2}(M_1^2 + M_2^2 + M_3^2) + \frac{c}{(\gamma_1\gamma_2\gamma_3)^{2/3}} + \frac{a(\gamma_2^2 + \gamma_3^2)}{(\gamma_2^2 - \gamma_3^2)^2}, \quad (2.31)$$

and the additional integral the form

$$F_6 = (F_{3N} + F_a)^2 + 4 \frac{f(\gamma_2^2\theta + 2c\gamma_1^2)(\gamma_3^2\theta + 2c\gamma_1^2)}{\gamma_1^4}, \quad (2.32)$$

where

$$F_{3N} = N_1N_2N_3 - 2c(N_1 + N_2 + N_3), \quad f = 2 \frac{a(\gamma_1\gamma_2\gamma_3)^{2/3}\gamma_1^2}{(\gamma_2^2 - \gamma_3^2)^2},$$

$$F_a = N_1f, \quad \theta = N_2N_3 - 2c + f, \quad N_i = \sqrt[3]{\gamma_1\gamma_2\gamma_3} \frac{M_i}{\gamma_i}. \quad (2.33)$$

Similar to above, the transformation inverse to (2.16) implies the Liouville integrable system in \mathbb{R}^3 with the Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + \frac{c}{(x_1x_2x_3)^{2/3}} + \frac{a(x_2^2 + x_3^2)}{(x_2^2 - x_3^2)^2}. \quad (2.34)$$

This system admits two additional integrals, namely the Jacobi integral (2.7) and integral (2.32), where one should replace γ_i by x_i and express M_i through momentum p_i in the standard way $\mathbf{M} = \mathbf{r} \times \mathbf{p}$ due to the invariance of momentum w.r.t. substitution (2.7) (see equality (2.22)).

3. THE N -BODY PROBLEM WITH A HOMOGENEOUS POTENTIAL OF THE DEGREE -2 , DEPENDING ON MUTUAL DISTANCES

3.1. The Algebra of Integrals

Consider the N -body problem with an interaction potential, depending only on mutual distances that is also a homogeneous function of the degree -2 . Its Hamiltonian has the form

$$H = \frac{1}{2} \sum \frac{\mathbf{p}_i^2}{m_i} + U_{-2}(|\mathbf{r}_i - \mathbf{r}_j|). \quad (3.1)$$

A special case of this problem is the system of 3 bodies on a straight line which was shown to be integrable on the zero level of the energy in papers [68, 69]. We consider a general problem of motion of N bodies in the three-dimensional space \mathbb{R}^3 . In this case one gets the following theorem:

Theorem 8. *The Hamiltonian (3.1), corresponding to the N -body motion in \mathbb{R}^3 with a homogeneous potential of degree $\alpha = -2$, admits the following ten functionally independent autonomous integrals of motion*

$$\begin{aligned} \mathbf{P} &= \sum \mathbf{p}_i, & \mathbf{S} &= \mathbf{P} \sum_{i=1}^N (\mathbf{r}_i, \mathbf{p}_i) - 2H \sum_{i=1}^N m_i \mathbf{r}_i, \\ \mathbf{M} &= \sum_i \mathbf{r}_i \times \mathbf{p}_i, & J &= 2IH - \left(\sum_{i=1}^N (\mathbf{r}_i, \mathbf{p}_i) \right)^2. \end{aligned} \quad (3.2)$$

Proof. For the potential under consideration equations (1.1) admit the following nonautonomous integrals of motion

$$\begin{aligned} \mathbf{c} &= \mathbf{R} - \frac{\mathbf{P}}{\sum m_i} t, & a &= 2 \sum_{i=1}^N (\mathbf{r}_i, \mathbf{p}_i) - 4Ht, \\ b &= 2Ht^2 - 2 \sum_{i=1}^N (\mathbf{r}_i, \mathbf{p}_i) t + I. \end{aligned} \quad (3.3)$$

After excluding the time from different pairs of integrals a and c_i , one gets the following three new integrals

$$\mathbf{S} = \mathbf{P} \sum_{i=1}^N (\mathbf{r}_i, \mathbf{p}_i) - 2H \sum_{i=1}^N m_i \mathbf{r}_i \quad (3.4)$$

besides of (1.6) and (2.7).

Integrals \mathbf{Q} in (1.6) found above and \mathbf{P}, \mathbf{S} are not independent and are connected in the following way

$$\mathbf{P} \times \mathbf{S} = 2H \sum_{i=1}^N m_i \mathbf{Q}. \quad (3.5)$$

Thus integrals (3.2) are the most convenient choice of independent integrals. The absence of additional connections between these integrals can be easily shown by calculation the corresponding Jacobian. \square

The Poisson brackets of integrals (3.2) have the form

$$\begin{aligned} \{M_\mu, M_\nu\} &= \varepsilon_{\mu\nu\rho} M_\rho, \quad \{M_\mu, S_\nu\} = \varepsilon_{\mu\nu\rho} S_\rho, \quad \{M_\mu, P_\nu\} = \varepsilon_{\mu\nu\rho} P_\rho, \quad \{M_\mu, J\} = 0, \\ \{S_\mu, S_\nu\} &= S_\mu P_\nu - P_\mu S_\nu, \quad \{S_\mu, P_\nu\} = P_\mu P_\nu - 2H \left(\sum m_i \right) \delta_{\mu\nu}, \quad \{S_\mu, J\} = -2JP_\mu, \\ \{P_\mu, P_\nu\} &= 0, \quad \{P_\mu, J\} = 2S_\mu. \end{aligned} \quad (3.6)$$

It follows from (3.1) that the constructed algebra of integrals is quadratic. The rank of this Poisson structure equals eight, therefore there are two Casimir functions

$$\begin{aligned} K_1 &= (\mathbf{P} \times \mathbf{S} - 2H \left(\sum m_i \right) \mathbf{M})^2, \\ K_2 &= \mathbf{S}^2 + J\mathbf{P}^2 - 2(\mathbf{M}, \mathbf{P} \times \mathbf{S}) + 2H \left(\sum m_i \right) (\mathbf{M}^2 - J). \end{aligned} \quad (3.7)$$

This yields the following theorem

Theorem 9. *Equations of motion (1.1), corresponding to the N -body motion in \mathbb{R}^3 with a homogeneous potential of the pairwise interaction of degree $\alpha = -2$, admit the reduction by six degrees of freedom using integrals (3.2).*

To prove the theorem one should find six integrals in involution. The Casimir functions (3.7) are two of them. Other four integrals can be found by the Darboux–Weinstein theorem as Darboux coordinates on a level of the Casimir functions.

By similar calculations for \mathbb{R}^2 one can show that the plain N -body problem can be reduced by four degrees of freedom. The corresponding constructive reduction procedure for the three-body problem with the potential

$$U = \sum_{i < j = 1}^3 \frac{a_{ij}}{(\mathbf{r}_i - \mathbf{r}_j)^2}, \quad \mathbf{r}_i \in \mathbb{R}^2 \quad (3.8)$$

are in the appendix 2 of the present paper. There is shown also the nonintegrability of this system using the Poincaré cross section. Recall that earlier for this problem [70] the meromorphic nonintegrability was shown only for the case $m_1 \neq m_2 = m_3 = 1$, $a_{i,j} = 1$.

4. THE JACOBI PROBLEM ON A LINE

4.1. The Integrability and Superintegrability

As an example, consider the Jacobi problem on three-body motion on a line with the potential of the form

$$U = \sum_{i < j = 1}^3 \frac{a_{ij}}{(x_i - x_j)^2}. \quad (4.1)$$

The integrability of this problem for the particular case $a_{ij} = \gamma m_i m_j$ was shown by Jacobi in [32], where he found the separated variables. Later, the studying of the quantization problem [34] led L. Calogero to the problem of N -body motion on a line with potential (4.1). The integrability of this problem in the case of equal masses and constants of interaction was shown by J. Moser by constructing a Lax pair [36]. The integrability of similar problems with more complicated potentials of interaction, particularly connected with root systems of semisimple Lie algebras, was considered in [37, 71–75].

In the following the interest to this problem revived in connection with the problem of its superintegrability, i.e. the presence of a redundant number of first integrals. Note that lately the problem of superintegrability of Hamiltonian systems is actively discussed both by mathematicians and by physicists such as Ranada [76], Gonera [77], Wojciechowski [78], Winternitz [79], Kozlov and Fedorov [80]. It should be noted that many papers considerably overlap each other. This interest is generated by different applications to classical and quantum mechanics since closed orbits admit the simplest quantization. Recently there were found some superintegrable systems with quite complex integrals of motion. For example, one can mention superintegrable open Toda chains found by Damianou [81] and superintegrable systems, connected with scattering problems [63], though the importance of such superintegrability for the scattering process is not yet evident. At the end of the present section we shall formulate a hypothesis on a general form of superintegrable systems with integrals of an arbitrary high degree in momentum.

We can generalize the full set of integrals (3.2) for the case of arbitrary masses and constants of interaction in the following form

$$\begin{aligned}
 H &= \frac{1}{2} \sum_{i=1}^3 \frac{p_i^2}{m_i} + \sum_{i<j=1}^3 \frac{a_{ij}}{(x_i - x_j)^2}, \quad P = \sum_{i=1}^3 p_i, \\
 S &= P \sum_{i=1}^3 x_i p_i - 2H \sum_{i=1}^3 m_i x_i, \quad J = 2H \sum_{i=1}^3 m_i x_i^2 - \left(\sum_{i=1}^3 x_i p_i \right)^2.
 \end{aligned}
 \tag{4.2}$$

In the case $m_1 = m_2 = m_3 = 1$ and $a_{12} = a_{13} = a_{23} = a$ there is also another integral

$$F = p_1 p_2 p_3 - \sum_{i<j=1}^3 \frac{a p_k}{(x_i - x_j)^2},
 \tag{4.3}$$

cubic in momentum. Thus, in this case the three-body system is superintegrable or maximally superintegrable. Note that its superintegrability was claimed in [79] and its superseparability in [83] without using the integral of the third degree. However it was later mentioned in [84] that the superintegrability of this system can not be proved without this integral. Contrary to this case, for the Rosochatius system (2.26), which can be obtained from the present one using reduction by theorem 6, integrals quadratic in momentum are sufficient for its superintegrability.

Note also that cubic integrals (2.25), (4.3) and (2.29) for different systems are quite similar with each other. It would be interesting to find a universal form of such integral and a condition for the potential enables its existence.

4.2. Reduction to a System on the Sphere

Consider the system on the sphere \mathbb{S}^2 that is obtained from the Jacobi problem by transformation (2.16). As it was said above the integral J becomes the Hamiltonian of the reduced system on the sphere of the form

$$\tilde{H} = J/2 = \frac{1}{2} \mathbf{M}^2 + V(\gamma),
 \tag{4.4}$$

where

$$V(\gamma) = IU(\mathbf{x}) = \sum_{i<j=1}^3 \frac{a_{ij}^*}{(\gamma_i \sqrt{m_j} - \gamma_j \sqrt{m_i})^2}.
 \tag{4.5}$$

Other integrals become

$$\begin{aligned}
 H &= \frac{\tilde{H}}{I} + \frac{I'^2}{8I^3}, \quad P = \frac{1}{\sqrt{I}} \sum_{i=1}^3 \sqrt{m_i} \gamma'_i + \frac{I'}{2I^{3/2}} \sum_{i=1}^3 \sqrt{m_i} \gamma_i, \\
 S &= \frac{I'}{2I^{3/2}} \sum_{i=1}^3 \sqrt{m_i} \gamma'_i - \frac{2\tilde{H}}{\sqrt{I}} \sum_{i=1}^3 \sqrt{m_i} \gamma_i,
 \end{aligned}
 \tag{4.6}$$

where $f' = \frac{df}{d\tau}$.

Due to (4.6) after the variable change (2.16) integrals H , P and S are nonautonomous since they depend upon variables I and I' . The dependence of this integrals on time can be obtained by relation (2.4) and the change of time in (2.16). In the general case one can construct one nonautonomous integral

$$G = \frac{S^2 + JP^2}{2E} = 2\tilde{H} \left(\sum_{i=1}^3 \sqrt{m_i} \gamma_i \right)^2 + \left(\sum_{i=1}^3 \sqrt{m_i} \gamma'_i \right)^2
 \tag{4.7}$$

from integrals (4.6). Thus reduction (2.16) produces the integrable system on the sphere with the additional first integral (4.7). Moreover one can formulate the following proposition

Proposition 3. *Any system on the sphere with the Hamiltonian (4.4) and a homogeneous potential of the degree -2 depending on differences $(\gamma_i - \gamma_j)$, $i, j = 1 \dots 3$ is integrable and has additional integral (4.7).*

4.3. The Jacobi Problem as a Superposition of Hooks Centers on the Sphere and Its Generalizations

It was mentioned in 4.1 that in the case $m_1 = m_2 = m_3 = 1$ and $a_{12} = a_{13} = a_{23} = a$ the initial system has the additional integral of the third degree (4.3). In order to calculate the corresponding integral on the sphere consider the zero level of the integral P . On this level the radius-vector R of the mass center becomes the first integral and substitution (2.16) leads to

$$R = \text{const} = \frac{\sqrt{I}}{\sum_{i=1}^3 m_i} \sum_{i=1}^3 \sqrt{m_i} \gamma_i. \quad (4.8)$$

By calculating derivative of (4.8) w.r.t. the new time one gets the dependence of I and I' on the new time

$$\frac{1}{\sqrt{I}} = \frac{\sum_{i=1}^3 \sqrt{m_i} \gamma_i}{R \sum_{i=1}^3 m_i}, \quad \frac{I'}{I^{3/2}} = -2 \frac{\sum_{i=1}^3 \sqrt{m_i} \gamma_i'}{R \sum_{i=1}^3 m_i}. \quad (4.9)$$

The substitution of these expressions into the integral F written via new variables leads to

$$F = \prod_{i=1}^3 \sum_{j=1}^3 (\gamma_i' \gamma_j - \gamma_i \gamma_j') - a \left(\sum_{i=1}^3 \gamma_i \right)^2 \sum_{i>j, k \neq i, j}^3 \sum_{l=1}^3 \frac{(\gamma_k' \gamma_l - \gamma_k \gamma_l')}{(\gamma_i - \gamma_j)^2} \quad (4.10)$$

up to constants.

Consider the reduced system on the sphere in the case of equal masses $m_1 = m_2 = m_3 = 1$ and constants of interaction $a_{12} = a_{13} = a_{23} = a$. To do so we rotate the coordinate system to make vector $(1, 1, 1)$ a vertical one

$$\tilde{\gamma}_1 = \frac{1}{\sqrt{2}}(\gamma_1 - \gamma_2), \quad \tilde{\gamma}_2 = \frac{1}{\sqrt{6}}(\gamma_1 + \gamma_2 - 2\gamma_3), \quad \tilde{\gamma}_3 = \frac{1}{\sqrt{3}}(\gamma_1 + \gamma_2 + \gamma_3). \quad (4.11)$$

By omitting tildes one gets the Hamiltonian in new variables in the form

$$H = \frac{1}{2} \mathbf{M}^2 + \frac{a}{2} \left(\sum_{i=1}^3 \frac{1}{(\mathbf{r}_i, \boldsymbol{\gamma})^2} \right), \quad (4.12)$$

where

$$\mathbf{r}_1 = (1, 0, 0), \quad \mathbf{r}_2 = (1/2, -\sqrt{3}/2, 0), \quad \mathbf{r}_3 = (-1/2, -\sqrt{3}/2, 0). \quad (4.13)$$

Due to (4.12) the Jacobi problem describes the particle motion on the sphere in the field of three identical Hook centers lying on the equator in vertexes of an equilateral triangle. By a Hook center on the sphere we mean the potential $\frac{1}{(\mathbf{r}_i, \boldsymbol{\gamma})^2}$, which is proportional to the square of the tangent of the angle distance between the particle and the center.

A more general case of the particle motion on the sphere in the field of N Hook centers lying on one equator was considered in [20]. Particularly there was shown the integrability of this problem for an arbitrary number of Hook centers of different intensity and found an additional first integral of motion being the particular case of integral (4.7).

The following hypothesis arises in connection with the above mentioned analogy between the superintegrable Jacobi problem and a particle motion in the field of three Hook centers.

Conjecture 1. *The problem on a particle motion in the field of $2N + 1$ identical Hook centers in $(2N + 1)$ vertexes of a regular polyhedron admits two independent first integrals of motion and is superintegrable. The first integral is given by (4.7) and the second one is a polynomial of $(2N + 1)$ degree in momentum.*

To confirm this hypothesis we made some numerical experiments. Their results prove with a quite credibility the accuracy of this hypothesis for five and seven Hook centers.

4.4. The Euler–Jacobi Problem

There is one more interesting generalization of the Jacobi problem. A many-body system with the Hamiltonian

$$H = \frac{1}{2} \sum_{i=1}^N p_i^2 + \sum_{i<j=1}^N \frac{\mu_{ij}^2}{(x_i - x_j)^2} \tag{4.14}$$

was considered in [85], where constants of interaction μ_{ij} are now dynamic variables and the corresponding Poisson structure for $N = 3$ has the form

$$\begin{aligned} \{\mu_i, \mu_j\} &= e_{ijk}\mu_k, & \{\mu_i, x_j\} &= 0, & \{\mu_i, p_j\} &= 0, \\ \{x_i, p_j\} &= \delta_{ij}, & \{p_i, p_j\} &= 0, & \{x_i, x_j\} &= 0, \end{aligned} \tag{4.15}$$

where we use the usual notations $\mu_{ij} = e_{ijk}\mu_k$. J. Gibbons and T. Hermsen [85] proved the integrability of this system, finding for it the Lax representation. Later S. Wojciechowski showed its superintegrability [86]. The R -matrix formalism for this problem and some of its generalizations was constructed in [87–90]. In the following the similar introduction of an additional internal degree of freedom was made for other generalizations of the Calogero–Moser system, including potentials, connected with root systems of semisimple Lie algebras. The complete description of these results can be found in [91]. It should be noted also that a particular case of this potential connected with the algebra D_3 was considered in [92, 93] in connection with studying of the four dimensional $SU(2)$ Yang–Mills theory under the assumption of a space homogeneity of a gauge field.

It is not difficult to show that the presence of additional degrees of freedom in system (4.14) does not prevent its reduction by theorem 6 from section 2.2. Thus using this theorem we get a new superintegrable system with the Hamiltonian

$$H = \frac{1}{2} \mathbf{M}^2 + \sum_{i<j=1}^3 \frac{\mu_{ij}^2}{(\gamma_i - \gamma_j)^2}, \tag{4.16}$$

defined on the direct product $e(3) \times so(3)$ with Poisson brackets

$$\begin{aligned} \{M_i, M_j\} &= e_{ijk}M_k, & \{M_i, \gamma_j\} &= e_{ijk}\gamma_k, & \{M_i, \mu_j\} &= 0, \\ \{\mu_i, \mu_j\} &= e_{ijk}\mu_k, & \{\mu_i, \gamma_j\} &= 0, & \{\gamma_i, \gamma_j\} &= 0. \end{aligned} \tag{4.17}$$

Consider the following generalization of system (4.16) including constants a_{ij} in the potential of interaction

$$H = \frac{1}{2} \mathbf{M}^2 + \sum_{i<j=1}^3 \frac{a_{ij}\mu_{ij}^2}{(\gamma_i - \gamma_j)^2}. \tag{4.18}$$

System (4.18) has three Casimir functions γ^2 , (\mathbf{M}, γ) , μ^2 and at least one integral of motion (4.7). This allows the reduction of (4.18) to a system with two degrees of freedom. It is natural to suppose that some combinations of three integrals of system (4.14) also can be integrals for system (4.18), though this question now is open.

5. THE JACOBI PROBLEM ON A PLANE

Consider the three-body problem on a plane with the Hamiltonian

$$H = \frac{1}{2} \sum_{i=1}^3 \frac{\mathbf{p}_i^2}{m_i} + \sum_{i>j} \frac{a_{ij}}{(\mathbf{r}_i - \mathbf{r}_j)^2}, \quad \mathbf{r}_i, \mathbf{p}_i \in \mathbb{R}^2. \tag{5.1}$$

According to section 3, this system has six first integrals of motion

$$\begin{aligned} \mathbf{P} &= \sum_{i=1}^3 \mathbf{p}_i, \quad \mathbf{S} = \mathbf{P} \sum_{i=1}^3 (\mathbf{r}_i, \mathbf{p}_i) - 2H \sum_{i=1}^3 m_i \mathbf{r}_i, \\ M &= \sum_{i=1}^3 (x_i p_{y_i} - y_i p_{x_i}), \quad J = 2IH - \left(\sum_{i=1}^3 (\mathbf{r}_i, \mathbf{p}_i) \right)^2, \end{aligned} \quad (5.2)$$

where $I = \sum_{i=1}^3 m_i \mathbf{r}_i^2$ is the central moment of inertia for of this system. Let us prove the following theorem on reduction of this system.

Theorem 10. *System (5.1) allows the reduction by four degrees of freedom using integrals (5.2). The Hamiltonian of the reduced system has the form*

$$\begin{aligned} H &= \frac{1}{2} p_\theta^2 + \frac{1}{2} \frac{(p_\psi + M)^2}{\sin^2 \theta} + \frac{1}{2} \frac{(p_\psi - M)^2}{\cos^2 \theta} + \frac{\mu_1 a_{12}}{\cos^2 \theta} \\ &+ \frac{\mu_2 m_1^2 a_{13}}{m_1^2 \sin^2 \theta + \mu_1 \mu_2 \cos^2 \theta + m_1 \sqrt{\mu_1 \mu_2} \sin 2\theta \cos \psi} \\ &+ \frac{\mu_2 m_2^2 a_{23}}{m_2^2 \sin^2 \theta + \mu_1 \mu_2 \cos^2 \theta - m_2 \sqrt{\mu_1 \mu_2} \sin 2\theta \cos \psi}, \end{aligned} \quad (5.3)$$

where $\theta \in (0, \frac{\pi}{2})$, $\psi \in [0, \pi)$, $\mu_1 = \frac{m_1 m_2}{m_1 + m_2}$, $\mu_2 = \frac{(m_1 + m_2) m_3}{m_1 + m_2 + m_3}$.

Proof. For the proof of the theorem we shall make the following row of transformations

1. The reduction using integrals \mathbf{P} and \mathbf{S} can be done by the classical transition to barycentric coordinates (Jacobi coordinates)

$$\tilde{\mathbf{r}}_i = \mathbf{r}_{i+1} - \mathbf{R}_i, \quad \text{where} \quad \mathbf{R}_i = \sum_{k=1}^i m_k \mathbf{r}_k / \sum_{k=1}^i m_k, \quad i = 1, \dots, 2.$$

2. The reduction using the integral of the angular momentum M can be done in the following way. Let us introduce polar coordinates $\tilde{\mathbf{r}}_i = (\rho_i \cos \varphi_i, \rho_i \sin \varphi_i)$ and make the following change of variables $\varphi = \varphi_1 + \varphi_2$, $\psi = \varphi_1 - \varphi_2$. Here φ is the cyclic variable, corresponding to integral $M = p_\varphi$. By excluding φ we get the reduced system.
3. The reduction using the Jacobi integral J can be done by applying theorem 6 from section 2.2 to variables ρ_1, ρ_2 and making the following change of variables and the time

$$\rho_1 = \sqrt{\frac{I}{\mu_1}} \cos \theta, \quad \rho_2 = \sqrt{\frac{I}{\mu_2}} \sin \theta, \quad dt = I d\tau.$$

By explicit calculations one can show that after these transformation we obtain finally the Hamiltonian system with two degrees of freedom and Hamiltonian (5.3). \square

Now consider the problem of integrability of system (5.3) with two degrees of freedom. Fig. 1 represents the corresponding Poincaré cross section by the plane $\theta = \frac{\pi}{4}$. Even for equal masses $m_i = 1$ and equal constants of interaction $a_{ij} = 1$ on the zero level of momentum $M = 0$ the phase portrait contains a chaotic layer (5.3). Thus the constructed Poincaré cross-section is a computer proof of the nonintegrability of system (5.3).

The Poincaré cross section for system (5.3) by the plane $\theta = \frac{\pi}{4}$ for $m_1 = m_2 = m_3 = a_{12} = a_{23} = a_{13} = 1$ on the zero level of momentum $M = 0$ and the energy level $H = 3.8$.]

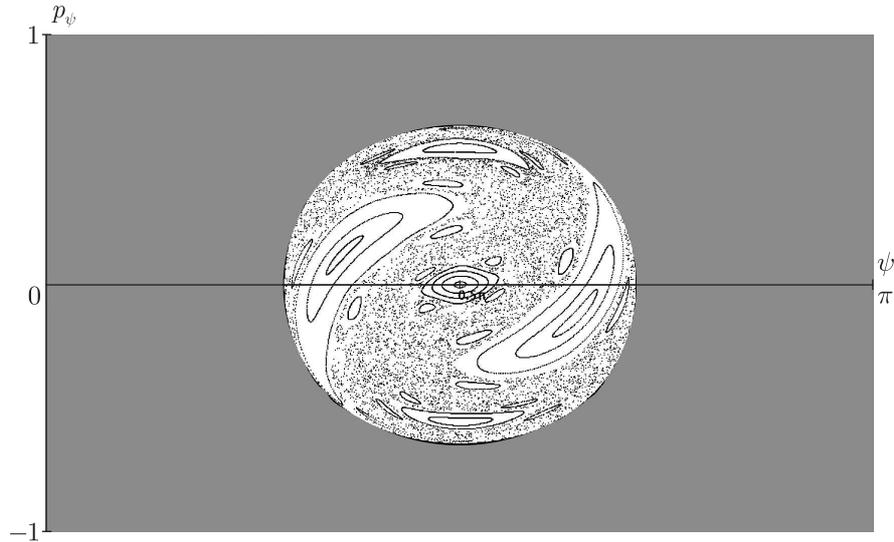


Fig. 1.

6. THE GENERALIZATION OF THE LAGRANGE IDENTITY AND THE JACOBI INTEGRAL

Consider the Hamiltonian system with the following Hamiltonian

$$H = \frac{1}{2} \sum \frac{\mathbf{p}_i^2}{m_i} + U_{-2}(|\mathbf{r}_i - \mathbf{r}_j|) + V(I, \mathbf{R}), \quad \mathbf{r}_i, \mathbf{p}_i \in \mathbb{R}^n, \tag{6.1}$$

where $U_{-2}(|\mathbf{r}_i - \mathbf{r}_j|)$ is a homogeneous function of the degree $\alpha = -2$, depending only on mutual distances $|\mathbf{r}_i - \mathbf{r}_j|$, $i, j = 1, \dots, N$, and $V(I, \mathbf{R})$ is an arbitrary function depending on the moment of inertia I and the radius vector of the mass center \mathbf{R} . The evolution equations for I and \mathbf{R} have the form

$$\begin{cases} \ddot{I} = 4(H - V) - 4I \frac{\partial V}{\partial I} - 2(\mathbf{R}, \frac{\partial V}{\partial \mathbf{R}}), \\ \ddot{\mathbf{R}} = -2 \frac{\partial V}{\partial I} \mathbf{R} - \frac{1}{\mu} \frac{\partial V}{\partial \mathbf{R}}, \end{cases} \tag{6.2}$$

where $\mu = \sum m_i$ is the total mass of the system. These equations generalize the Lagrange identity (2.3). Equations (6.2) are separated from the whole system and can be integrated independently. The following theorem describes some particular cases of the function $f(I, \mathbf{R})$, when system (6.2) (and so the whole system) admits additional integrals of motion.

Theorem 11. *System (6.1) admits additional integrals of motion in the following cases*

1. For $V(I, \mathbf{R}) = f(I)$ system (6.1) admits additional integrals

$$\begin{aligned} J &= 4(H - f)I - \frac{1}{2} \dot{I}^2, \\ G_i &= IP_i^2 - \mu P_i R_i \dot{I} + 2\mu^2 R_i^2 (H - f), \quad i = 1, \dots, n, \end{aligned} \tag{6.3}$$

where $\mathbf{P} = \mu \dot{\mathbf{R}}$.

2. For $V(I, \mathbf{R}) = kI + f(\mathbf{R})$, where k is an arbitrary parameter, system (6.1) admits an additional integral

$$F = \frac{1}{2} \mathbf{P}^2 + k\mathbf{R}^2 + \frac{1}{\mu} f(\mathbf{R}). \tag{6.4}$$

3. For $V(I, \mathbf{R}) = f(x) + g(\mathbf{R})$ (here $x = \mu I - \mu^2 \mathbf{R}^2$ is a new variable) system (6.1) admits additional integrals

$$\begin{aligned} F_1 &= \frac{1}{2} \dot{\mathbf{R}}^2 + \frac{1}{\mu} g(\mathbf{R}), \\ F_2 &= \frac{1}{2} \dot{x}^2 - 4\mu x(H - f(x) - \mu F_1). \end{aligned} \quad (6.5)$$

Proof.

1. The existence of the first integral (6.3) follows from theorem 5 of section 2 in the present paper. System (6.2) is divided into the following pairs of equations

$$\begin{cases} \ddot{I} = 4(H - f) - 4I \frac{\partial f}{\partial I}, \\ \ddot{R}_i = -2 \frac{\partial f}{\partial I} R_i, \end{cases} \quad (6.6)$$

having integrals G_i , which are analogs of the integral (4.7).

2. In this case the second equation from (6.2) has the form of the Newton law

$$\ddot{\mathbf{R}} = -2k\mathbf{R} - \frac{1}{\mu} \frac{\partial f}{\partial \mathbf{R}} \quad (6.7)$$

and admit evident energy integral (6.4).

3. One can express equation (6.2) through variables x, \mathbf{R}

$$\begin{cases} \ddot{x} = 4\mu(H - f(x) - g(\mathbf{R})) - 4\mu x \frac{\partial f}{\partial x} - 2\mu^2 \dot{\mathbf{R}}^2, \\ \ddot{\mathbf{R}} = -\frac{1}{\mu} \frac{\partial g}{\partial \mathbf{R}}. \end{cases} \quad (6.8)$$

Similar to the previous section, the second equation admits the energy integral

$$F_1 = \frac{1}{2} \dot{\mathbf{R}}^2 + \frac{1}{\mu} g(\mathbf{R}). \quad (6.9)$$

One can express $\dot{\mathbf{R}}^2$ through the integral F_1 and substitute the result into the first equation in (6.8). This yields the equation of the Newton type

$$\ddot{x} = 4\mu(H - f(x) - x \frac{\partial f}{\partial x} - \mu F_1), \quad (6.10)$$

which admits the following first integral

$$F_2 = \frac{1}{2} \dot{x}^2 - 4\mu x(H - f(x) - \mu F_1). \quad (6.11)$$

□

Below are some results concerning the integrability of N -body problem on a line, which follow from theorem 11.

Corollary 2. *The three-body problem on a line with the potential $U = U_{-2}(|x_i - x_j|) + f(I)$ admits two additional integrals of motion and is integrable.*

Corollary 3. *The two-body problem on a line with the potential $U = U_{-2}(|x_1 - x_2|) + kI + f(\mathbf{R})$ admits an additional integrals of motion and is integrable.*

Corollary 4. *The three-body problem on a line with the potential $U = U_{-2}(|x_i - x_j|) + f(x) + g(\mathbf{R})$ admits two additional integrals of motion and is integrable.*

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