

# On the Motion of a Body with a Rigid Shell and Variable Mass Geometry in a Perfect Fluid

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## 1. MOTION OF A VARIABLE BODY

We consider the problem of a body with a rigid shell moving in an infinite volume of a homogeneous perfect fluid. The fluid motion and the fluid in itself are assumed, respectively to be nonrotational and quiescent at infinity. However, in contrast to the classical formulation of the problem on the body motion, we consider that the mass geometry of the body can vary under the action of internal forces according to a law known *a priori*. For example, a material point moves inside a case in accordance with a given law.

We assume that the body and the fluid were immobile at the initial time moment. It is of interest whether or not it is possible to displace the body's case from the given position to an arbitrary preassigned position by means of an appropriate variation of body-mass geometry (under the action of internal forces). At first glance, this seems to be impossible, because the center of mass for the body–fluid system is at rest. However, this argument cannot be taken into account. Firstly, the center of mass of an infinite volume of fluid is uncertain; secondly, we are interested only in the displacement of the body's case but not of the fluid.

We formulate our principal result for a body whose boundary has three mutually orthogonal planes of symmetry. It turns out that, if not all the associated masses of the body (depending only on the body shape) are equal, a driving force can be produced inside its case due to the displacements of points. Moreover, the body can be displaced from an arbitrary position to another arbitrary position in the case of an appropriate control of the mass geometry. We note that the property of the total body controllability is lost if all the associated masses coincide.

The problem under consideration is a particular case of a more general problem on the motion of a deformable body in a fluid, which is of a substantial importance in studying the mechanism of swimming fish and also the cavitation phenomenon. The first results in this field were obtained by Taylor [1] and Lighthill [2].

They considered motion in a viscous fluid. The body–fluid energy exchange takes place because of the vortex separation from sharp edges of the body and also due to an inertial action of the fluid onto the body. In [3], the model problem on dynamics of a deformable plate in a fluid was considered. The proper motion of the plate represents a running wave. An estimate of the effect of wave parameters on the value of a driving-force was made. A qualitative explanation of the mechanism for the motion of fish on the basis of a model of motion in a solid channel was given in [4].

In this connection, the question arises as to whether the motion of the body is possible at the expense of deforming its own boundary in a perfect fluid performing a nonrotational motion. The positive solution to this problem is given in [5]. A similar problem was also considered in [6]. The approach used in [5] was repeated in [7]. In [8], the possibility of producing a driving force was shown for the case when a variable plate of infinite length moves in a perfect nonrotational fluid. This problem was considered most rigorously in [9]. It is assumed that just as after the deformation (without the fluid), the center-of-mass position and that of the principal axes are invariable. The force and the moment acting from the fluid onto the body can be found using the generalized Lagally theorem [10].

In our study, we investigate a more complicated problem on the possibility of the gradual motion of a body with a rigid boundary in a perfect fluid without vortices.

## 2. THEOREM OF CONTROLLABILITY

Thus, we consider the motion of a body with three orthogonal planes of symmetry. Inside the body, a material point of mass  $m$  can move (under the action of internal forces). We relate to the body a mobile reference system  $O\xi\eta\zeta$  so that the kinetic energy of the body–fluid system can be represented in the form

$$T' = \frac{(A\mathbf{v}, \mathbf{v})}{2} + \frac{(C\boldsymbol{\omega}, \boldsymbol{\omega})}{2}.$$

Here,  $\mathbf{v}$  is the velocity of the point  $O$ ,  $\boldsymbol{\omega}$  is the angular velocity of the body with respect to the mobile axes,

$A = \text{diag}(a_1, a_2, a_3)$ , and  $C = \text{diag}(c_1, c_2, c_3)$ ; all the constants  $a_k$  and  $c_k$  are positive.

The motion of the point with mass  $m$  is assigned by the certain known functions  $\xi(t)$ ,  $\eta(t)$ , and  $\zeta(t)$ . The components of the absolute velocity of this point in the mobile axes  $\xi\eta\zeta$  have the form

$$\begin{aligned} u_1 &= v_1 + \dot{\xi} - \omega_3\eta + \omega_2\zeta, \\ u_2 &= v_2 + \dot{\eta} + \omega_3\xi - \omega_1\zeta, \\ u_3 &= v_3 + \dot{\zeta} + \omega_1\eta - \omega_2\xi. \end{aligned}$$

Here,  $v_1, v_2, v_3(\omega_1, \omega_2, \omega_3)$  are the components of the vector  $v(\omega)$ .

The total kinetic energy of a variable body is

$$T = T' + \frac{m(u_1^2 + u_2^2 + u_3^2)}{2}.$$

The theorems on the variation of the momentum and the angular momentum with respect to the mobile axes yield the generalized Kirchhoff equations

$$\left(\frac{\partial T}{\partial \omega}\right) \cdot + \left[\omega, \frac{\partial T}{\partial \omega}\right] + \left[v, \frac{\partial T}{\partial v}\right] = 0, \tag{2.1}$$

$$\left(\frac{\partial T}{\partial v}\right) \cdot + \left[\omega, \frac{\partial T}{\partial v}\right] = 0.$$

According to [9], this type of equation also describes the motion of a body with a variable boundary. In [5–7], equations (2.1) were not used explicitly.

Equations (2.1) need to be supplemented by kinematic relations. Let  $\theta, \varphi$ , and  $\psi$  be the Euler angles defining the orientation of a mobile trihedron ( $\xi\eta\zeta$ ) with respect to the immobile system ( $xyz$ ). Let  $x, y$ , and  $z$  also be the coordinates of the point  $O$ . We now use the kinematic Euler formulas

$$(\dot{\theta}, \dot{\psi}, \dot{\varphi})^T = B(p, q, r)^T \tag{2.2}$$

and the formulas for the passage from a mobile trihedron to the immobile one

$$(\dot{x}, \dot{y}, \dot{z})^T = D(v_1, v_2, v_3)^T. \tag{2.3}$$

The expressions for the elements of matrices  $B$  and  $D$  ( $D$  is the orthogonal matrix) in terms of the Euler angles are well known (see, for example, [11]). Equations (2.1)–(2.3) represent the total set of equations of motion for the system under consideration.

We assume that the system begins to move from the quiescent state. In this case, the integrals of momentum and of the angular momentum take the form

$$\frac{\partial T}{\partial v} = \frac{\partial T}{\partial \omega} = 0.$$

By virtue of the positive definiteness of the form  $T'$ , these equations can be solved with respect to  $v$  and  $\omega$ .

Substituting the obtained expressions into kinematic relationships (2.2) and (2.3), we arrive at

$$(\dot{x}, \dot{y}, \dot{z}, \dot{\theta}, \dot{\varphi}, \dot{\psi})^T = U\dot{\xi} + V\dot{\eta} + W\dot{\zeta}. \tag{2.4}$$

The components of the six-dimensional vectors  $U, V$ , and  $W$  depend parametrically on the functions  $\xi(t), \eta(t), \zeta(t)$ , and the Euler angles, as well as on the coefficients of the form  $T'$ . Equations (2.4) represent a closed set of first-order differential equations on the  $e(3)$  group of motions in three-dimensional space. The functions  $\xi, \eta$ , and  $\zeta$  serve as controls. The position of a body is determined by six parameters  $(x, y, z, \theta, \varphi, \psi) = z, z \in e(3)$ .

**Remark.** The indicated reduction is an example of a more general construction for mechanical systems with a configuration space in the form of the Lie group and the left-invariant kinetic energy. If the motion proceeds without the action of external forces, the role of integrals for the momentum and for the angular momentum is played by the total set of Noether integrals [12].

The system under consideration is called completely controllable if, for an arbitrary  $\varepsilon > 0$  and two arbitrary positions  $z_1$  and  $z_2$  of the body, it is possible to find the piecewise smooth functions  $\xi(t), \eta(t)$ , and  $\zeta(t)$  with  $t_1 \leq t \leq t_2$  such that

$$\begin{aligned} |\xi(t)| \leq \varepsilon, \quad |\eta(t)| \leq \varepsilon, \quad |\zeta(t)| \leq \varepsilon, \\ z(t_1) = z_1, \quad z(t_2) = z_2. \end{aligned}$$

The time  $t_2 - t_1$  of the motion substantially depends on the parameter  $\varepsilon$ .

**Theorem.** *The system is quite controllable if and only if not all the associated masses  $a_1, a_2$ , and  $a_3$  are equal to each other.*

Indeed,  $x, y$ , and  $z$  are limited as functions of time if  $a_1 = a_2 = a_3$ . In turn, this fact follows from the immobility of the center of mass for the system of the body–fluid–point of the integral in equations (2.4):

$$(a + m)(x, y, z)^T + mD(\xi, \eta, \zeta)^T = \text{const}.$$

For proving the sufficiency, we introduce the widened nine-dimensional space  $M$  with the coordinates  $x, y, z, \theta, \varphi, \psi, \xi, \eta$ , and  $\zeta$ .

Equations (2.4) determine the distribution of three-dimensional tangential planes. We also introduce three independent admissible vector fields  $V_1, V_2$ , and  $V_3$  with the components  $(U, 1, 0, 0), (V, 0, 1, 0)$ , and  $(W, 0, 0, 1)$ , respectively. Following the Rashevskii–Chow approach [13], we consider nine vector fields

$$\begin{aligned} V_1, V_2, V_3, [V_1, V_2], [V_1, V_3], [V_2, V_3], \\ [V_1, [V_1, V_2]], [V_3, [V_1, V_3]], [V_2, [V_2, V_3]], \end{aligned} \tag{2.5}$$

where  $[ , ]$  is the Jacobi bracket. If  $a_2 \neq a_3$ , for small values of  $\xi, \eta$ , and  $\zeta$ , these vectors turn out to be linearly independent at each point of  $e(3)$ .

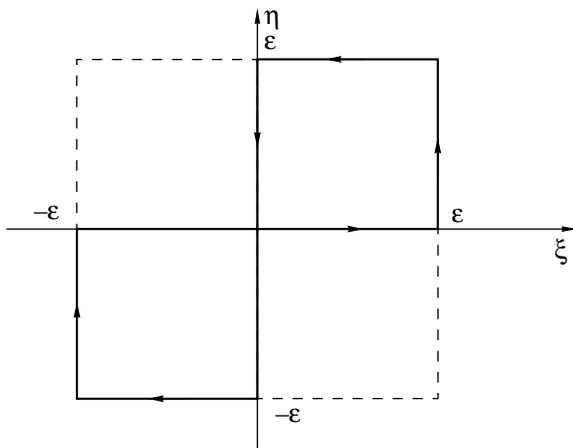


Figure.

Indeed, for  $\xi = \eta = \zeta = 0$ , the determinant of the matrix  $9 \times 9$  composed of the components of vectors (2.5) is equal to

$$\frac{(-2a_3^2m - 3a_3^2a_2 + a_2^2m + 3a_3a_2^2 + a_3a_2m)P(a_1, a_2, a_3)}{Q(a_1, a_2, a_3)c_1^2c_2^2c_3^2\sin\theta}.$$

Here,  $P(a_1, a_2, a_3)$  and  $Q(a_1, a_2, a_3)$  are polynomials with positive coefficients. Since  $a_k > 0$ , this expression vanishes only for  $a_2 = a_3$ .

If commutators (2.5) are chosen in a somewhat different way, the independence condition  $a_2 \neq a_3$  transforms into the condition  $a_1 \neq a_2$ , or  $a_1 \neq a_3$ .

It remains to utilize the Rashevskii–Chow theorem, according to which two arbitrary points of the connected domain in  $M$  given by the inequalities  $|\xi| \leq \epsilon$ ,  $|\eta| \leq \epsilon$ , and  $|\zeta| \leq \epsilon$  ( $\epsilon$  is small) can be connected by the piecewise smooth curve composed of segments from the integral curvilinear fields  $V_1, V_2$ , and  $V_3$ .

### 3. THE GUARANTEED CONTROL

We indicate an explicit method of controlling the mass geometry of a body inside a rigid shell, which makes it possible to transfer the body with unequal associated masses from one position to another arbitrary position. For this purpose, we initially consider an auxiliary problem on a gradual plane-parallel body's motion, when one of the symmetry planes (for example,  $\zeta = 0$ ) always remains invariable with time. The material point  $m$  also moves in this plane.

The group  $e(2)$  serves as a configuration space; the generalized coordinates are  $x, y$ , i.e., the coordinates of the body's point  $O$  and the rotation angle  $\phi$ . To simplify the calculation, we consider the limiting case as  $a_1 \rightarrow 0, a_2 \rightarrow \infty$ , and  $c_3$  tends to a finite limit. The possibility of such a passage to the limit was substantiated in [14]. In this case, equations (2.4) take the following form:

$$\begin{aligned} \dot{x} &= v \cos \phi, & \dot{y} &= v \sin \phi, & \dot{\phi} &= \omega, \\ v &= -\left(\dot{\xi} + \frac{\kappa \xi \dot{\eta}}{1 + \kappa \xi^2}\right), & \omega &= -\frac{\kappa \xi \dot{\eta}}{1 + \kappa \xi^2}, \end{aligned} \tag{3.1}$$

where  $\kappa = \frac{m}{c_3}$ .

Let the point  $m$  move with a constant (in its modulus) velocity along a closed curve shown in the figure. Using (3.1), we can calculate the increment of the coordinates for a period:

$$\begin{aligned} \Delta x &= 0, & \Delta y &= -2\epsilon \left(1 + \sin \mu - \frac{\sin \mu}{\mu}\right), \\ \Delta \phi &= 0. \end{aligned} \tag{3.2}$$

Here,  $\mu = \frac{\kappa \epsilon^2}{1 + \kappa \epsilon^2}$ . For small  $\epsilon \neq 0$ , it is evident that  $\Delta y \neq 0$ . Thus, on the average, the body is displaced forward by its wide side. By virtue of analyticity, this conclusion is also retained for almost all values of the parameters.

**Remark.** For the first two equations of set (3.1), we can see that motion of the body is subjected to a nonintegrable constraint  $\dot{x} \sin \phi - \dot{y} \cos \phi = 0$ . However, this motion proceeds not in accordance with laws of nonholonomic mechanics but according to the principles of vaconomic mechanics developed in [14].

Let the body occupy a given position at the time moment of  $t_1$ , and its center of mass coincide with the geometric center (the point  $O$ ); in particular, let the material point  $m$  for  $t = t_1$  be at the point  $O_1 = O$ . Using rotations of symmetric flywheels (gyrodines), the body can be turned around the point  $O_1$  and guided to an arbitrary preassigned position. This problem is well studied from various standpoints. If  $a_1 \neq a_2$ , we turn the body in order for the plane of symmetry  $\zeta = 0$  to involve a preassigned point  $O_2$  at which the body's center must be at the time moment  $t_2$ .

We now use formulas (3.2) valid for the plane-parallel motion. It is clear that  $\Delta y \rightarrow 0$  (as  $\epsilon^3$ ) when  $\epsilon \rightarrow 0$ . Hence, selecting a small  $\epsilon$ , it is possible to attain the situation when the spacing between the points  $O_1$  and  $O_2$  equals  $n\Delta y$ , where  $n$  is a certain integer. Thus, if the point  $m$  starting from the point  $O$  makes exactly  $n$  rotations along the eight-shaped curve shown in the figure, then the body's center occupies the position  $O_2$ . In this case, the point  $m$  turns out anew at the point  $O$ .

After this, it is necessary once more (for example, using symmetric flywheels) to turn the body around the point  $O_2$  and to orient the body in a preassigned way.

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