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Alexey V. Borisov

27.03.1965–24.01.2021

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Transformations of Lagrangians and closed orbits

Alain Albouy[1] and Lei Zhao[2]

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Let us call a natural Lagrangian a Lagrangian of the form $T + U$, where U is a function on a configuration space and T is a positive definite quadratic form on the tangent space. At the end of the XIX$^{\rm th}$ century, independent examples related to the Kepler problem, found by Halphen [7] and Goursat [6], led to the study of the “Transformation of the equations of Dynamics" (see e.g. Painlevé [8]). Such transformation is essentially *a change of time in a natural Lagrangian that gives another natural Lagrangian*.

Independently, Darboux [4] generalized to the surfaces of revolution a famous question by Bertrand [2] about the central forces in the plane. He solved the “main part” of his question. His solution was later completed by the works of Zoll [10], and more recently by Zagryadskiĭ, Kudryavtseva and Fedoseev (see e.g. [9]).

However, the following observation escaped Darboux and his followers: the “main part” of the solution, that Darboux described by simple rational formulas, consists in the “transformations” of the Lagrangian of the Kepler problem. The transformations are Appell’s central projection [1] and Darboux’ inversion [5], which changes the Lagrangian $T+U$ into the Lagrangian $UT + 1/U$.

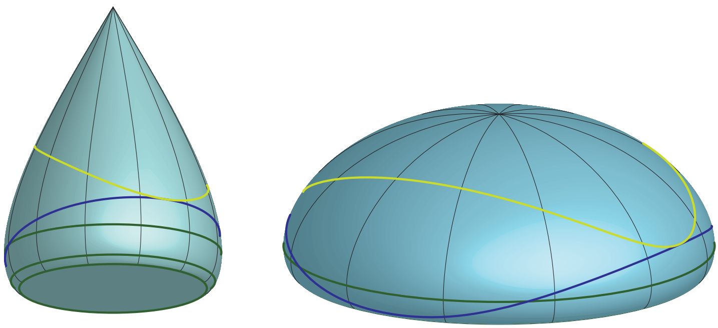

Fig. 1. Lagrangians with closed orbits on surfaces

Hopefully this result will attract again the attention on these “transformations”, which indeed are quite forgotten today (see however Borisov and Mamaev [3]).

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Multi-robot control strategies for monitoring a dynamically changing flood area

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This paper deals with utilizing multiple aerial robots to monitor a dynamically changing flood area. The problem requires developing a control strategy for the robots such that the motion of the complete flood area can be caged, tracked, and covered (see Fig. 1). The strategy consists of two stages: a caging stage and a covering stage. Correspondingly, the robots are divided into two groups: one for caging, referring as the boundary drones; the other for covering, referring as the inner drones. In the caging stage, boundary drones are uniformly distributed along the edge of the dynamic flood area, tracking its propagation with the use of a vision-based controller (based on the image segmentation).

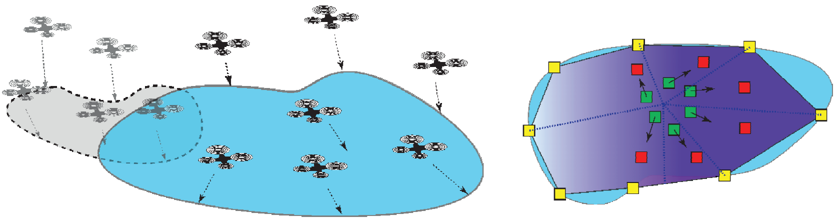

Fig. 1. Tracking a dynamically changing flood area (left) with the use of potential fields and centroidal Voronoi tessellation and density functions (right)

In the covering stage, inner drones are allocated in interior region of the flood zone, achieving an optimal coverage efficiency. An optimal configuration of the robots is generated with the use of Voronoi diagram over the coverage area, which maximizes the coverage efficiency by driving agents to the centroids in corresponding Voronoi cells. The construction of the Voronoi diagram takes into account possible obstacles. The relative importance of the region points is modeled by (possibly multiple) density functions. To address both adaptiveness and stability of the coverage control, a function approximation-based coverage controller has been developed [1]. The asymptotic stability of the controller was established, and its validity was demonstrated by simulations in ROS/Gazebo programming environment [2].

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KAM Tori in generic nearly-integrable Mechanical Systems

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We investigate the existence of Lagrangian invariant tori for nearly-integrable mechanical systems with Hamiltonian $\frac{1}{2}|y|^2 + \varepsilon f(x)$, f being a *generic* real-analytic potential; here $(y, x) \in \mathbb{R}^n \times \mathbb{T}^n$ are standard symplectic variables and the phase space is any bounded region times $\mathbb{T}^n$. The following statement hold:

Away from a neighbourhood $\mathcal{R}$ of double resonances, the phase space is filled with maximal KAM tori up to an exponentially small (*in* $1/\varepsilon^a$) *set. The "non perturbative set" $\mathcal{R}$ has a measure smaller than $\varepsilon|\log\varepsilon|^b$ for a suitable* $b > 0$.

In the talk I will discuss this result and sketch the mains steps needed to prove it.

Normal forms and averaging in an acceleration problem in nonholonomic mechanics

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We discuss unlimited growth of momentum in nonholonomic systems (Chaplygin's sleigh and Suslov's system) with periodically varying mass distribution. It is proved that, depending on the choice of the parameters, the momentum may grow as t^k with $k = 1, 2, 3$. The proof uses normal forms and averaging in a slightly unusual form. The talk is based on a joint work with I. Bizyaev and I. Mamaev.

A top on a vibrating base: new integrable problem of non-holonomic mechanics

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A spherical rigid body, rolling without sliding on a horizontal support, is considered. The body is axially symmetric but unbalanced (tip-top). The support performs high-frequency oscillations with small amplitude. To implement standard averaging procedure, we present equations of motion in quasi-coordinates in Hamiltonian form with additional terms of non-holonomy [1] and introduce new fast time variable. The averaged system is similar to the initial one with an additional term, known as vibrational potential [2–4]. This term depends on the single variable – the nutation angle, and according Chaplygin research [5], the averaged system is integrable. Some examples exhibit influence of vibrations on the dynamics.

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Energy transport in 1-D Hamiltonian lattices with applications to physics and engineering

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Centre of Integrable Systems, P.G. Demidov Yaroslavl State University, Yaroslavl, Russia

Talk presented at the International Conference "Regular and Chaotic Dynamics", November 22–December 3, Steklov Mathematical Institute, Moscow, dedicated to the memory of Professor Alexey Borisov.

Regular and chaotic dynamics of 1-D Hamiltonian lattices of N interacting particles has been extensively studied for more than 60 years, in view of its important applications to statistical mechanics and solid state physics [1]. Most studies have focused on *analytic* particle interactions, ranging from nearest neighbor to full range, often in the presence of on-site potentials [2]. Energy transport in such systems under periodic driving at one end of the lattice has revealed the important phenomenon of *supratransmission*, see e.g. [2, 3]. In the present lecture, I will first describe an approach from local to global dynamics in these systems as the total energy is increased. Next, I will apply this approach to 1-D Hamiltonian lattices that arise in mechanical engineering applications, such as graphene elasticity, Hollomon's law of "work hardening", under viscous or *hysteretic* damping. These involve nearest-neighbor interactions that are: (a) either purely non-analytic, (b) harmonic plus non-analytic or (c) analytic with non-analytic hysteretic damping effects [4, 5]. Finally, I will discuss energy transport in these systems, such as wave packet propagation and supratransmission, under periodic driving that includes additive noise effects [6].

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Libration points inside a spherical cavity in a uniformly rotating gravitating ball

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Relative equilibria of a material point in the field of attraction of a homogeneous ball with a spherical cavity are considered. It is assumed that the ball uniformly rotates around an axis perpendicular to the axis of symmetry of the body and passing through its center of mass. Families of relative equilibria (libration points) located inside the cavity both in the absence and in the presence of dry friction are studied. Stability and bifurcations of the found equilibria families are studied.

This investigation continues the cycle of studies devoted to the relative equilibria of a material point in the gravitational field of uniformly rotating objects. Studies of libration points located at a distance from rotating gravitating objects were undertaken in particular in publications of V. V. Beletsky & A.V. Rodnikov, and I.I. Kosenko (see, e.g., [1–4]). Relative equilibria that are not isolated due to the presence of dry friction were studied for model examples in [5,6] and for celestial bodies in [7,8].

General methods for investigating the existence and stability of equilibria in the presence of dry friction have been developed in [9–11]. The general theory of bifurcations of families of non-isolated equilibria has been developed in publications [12–15] (see also [16]).

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Quasi-periodic attractors for dissipative systems in Celestial Mechanics

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The existence of invariant tori using Kolmogorov – Arnold – Moser (KAM) theory has been proven in several models of Celestial Mechanics through dedicated analytical proofs combined with computer-assisted techniques. After reviewing some of such results, obtained in conservative frameworks, I present a recent result on the existence of invariant attractors for a dissipative model: the spin-orbit problem with tidal torque. This model belongs to the class of conformally symplectic systems, which are characterized by the property that they transform the symplectic form into a multiple of itself. Finding the solution of such systems requires to add a drift parameter. I will describe a KAM theorem for conformally symplectic systems in an a-posteriori format: assuming the existence of an approximate solution, satisfying the invariance equation up to an error term — small enough with respect to explicit condition numbers, — then we can prove the existence of a solution nearby. The theorem, which does not assume that the system is close to integrable, yields an efficient algorithm to construct invariant attractors for the spin-orbit problem for astronomically relevant values of the parameters. It also provides accurate estimates of the breakdown threshold of the invariant attractor. This talk refers to joint works [1–3] with R. Calleja, J. Gimeno, and R. de la Llave.

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Elliptic fixed points with an invariant foliation: some facts and more questions

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We address the following question: let $F : (R^2, 0) \to (R^2, 0)$ be a local analytic diffeomorphism defined in the neighborhood of the non-resonant elliptic fixed point 0 and let Ψ be a formal conjugacy to a normal form N. Supposing that F leaves invariant the foliation by circles centered at 0, what is the analytic nature of Ψ and N? The motivation comes from two examples of such local diffeomorphisms related to 1-parameter subfamilies of Arnold's family of analytic diffeomorphisms of the circle. An interesting technical feature is the use of results coming from the theory of holomorphic maps in one complex variable.

Resonance of ellipsoidal billiard trajectories and extremal rational functions

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We study resonant billiard trajectories within quadrics in the d-dimensional Euclidean space. We relate them to the theory of approximation, in particular the extremal rational functions on the systems of d intervals on the real line. This fruitful link enables us to prove fundamental properties of the billiard dynamics and to provide a comprehensive study of a large class of non-periodic trajectories of integrable billiards. A key ingredient is a functional-polynomial relation of a generalized Pell type. Applying further these ideas and techniques to s-weak billiard trajectories, we come to a functional-polynomial relation of the same generalized Pell type. This is a joint work with Milena Radnović.

Magnetic Chaplygin systems and generalized Demchenko case

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We describe the gyroscopic and the magnetic Chaplygin systems on fiber spaces and the reduction procedure for the corresponding G-Chaplygin systems. We derive the equations of motion of the reduced gyroscopic and magnetic G-Chaplygin systems. We introduce a problem of rolling of a ball with the gyroscope without slipping and twisting over a plane and a sphere in R^n, $n > 3$ as examples of magnetic $SO(n)$-Chaplygin systems. Specially the generalized Demchenko case in R^n is defined as the system (ball + gyroscope) without twisting when its inertia operator is $SO(n)$-invariant. The reduced system represents the magnetic geodesic flow on a sphere S^{n-1} endowed with the round-sphere metric, under the influence of the homogeneous magnetic field. We prove complete integrability of the generalized Demchenko system without twisting for $n = 3$ and $n = 4$, perform an explicit integration in elliptic functions, and provide the case study of the solutions in both cases.

The Lagrange Top and the general confluent Heun equation

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It is difficult to say anything new about the Lagrange top after the comprehensive analysis by Klein and Sommerfeld and the more modern and succinct treatment of Borisov and Mamaev. We would like to point out that the quantisation of the Lagrange top (after the addition of a quadratic potential) leads to the most general confluent Heun equation, also known as generalised spheroidal wave equation. In physics this equation is known as Teukolsky's master equation, which appears in the perturbation theory of Kerr black holes. We recall that the Lagrange top has two global S^1 symmetries generated by the corresponding action variables. The third, non-trivial action variable is not globally defined and exhibits Hamiltonian monodromy. We conclude that the generalised spheroidal wave equation exhibits quantum monodromy in its joint spectrum.

On the dynamics of a heavy ball that rolls on a rotating surface of revolution

Francesco Fassò

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We report on some results (obtained with M. Dalla Via, P.E. Petit Valdes Villareal and N. Sansonetto [1, 2]) on the dynamics of the class of nonholonomic mechanical systems formed by a heavy symmetric ball that rolls without sliding on a surface of revolution, which is either at rest or rotates about its (vertical) figure axis with uniform angular velocity Ω.

This is a subject whose first studies go back over a century and to which A. Borisov and his collaborators have given profound contributions.

In this talk, exploiting the existence of a conserved 'moving energy', we give conditions on the profile of the surface, and its rotational speed, that ensure the periodicity of the reduced dynamics and hence the quasi-periodicity of the unreduced dynamics on tori of dimension up to three. In particular, we show that the rotation of the surface has a 'stabilizing' effect on the dynamics.

Furthermore, we determine the equilibria of the reduced system, classifying them in three families, and determine their stability properties.

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A Darboux–Lax scheme for discrete integrable equations

Xenia Fisenko[1], Sotris Konstantinou-Rizos[1] and Pavlos Xenitidis[2]

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In this talk, we consider an important class of nonlinear difference equations on quad-graphs and present a novel method for constructing soliton solutions to them.

Quad-graph equations arise as the compatibility condition around the square of Darboux transformations for integrable nonlinear PDEs and serve as integrable discretisations of the latter. The importance of quad-graph equations is that one can construct solutions to them using simple algebraic schemes. Then, solutions to the associated nonlinear PDEs can be derived via continuum limits.

For those equations on quad-graphs which have the 3D-consistency property, one can algorithmically construct a Lax representation and also a Bäcklund transformation. However, not all quad-graph equations are 3D consistent. We present a new Darboux–Lax scheme for constructing solutions to equations on quad-graphs which have a Lax representation but are not necessarily 3D consistent.

Platonic solids and symmetric solutions of the N-vortex problem on the sphere

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We consider the N-vortex problem on the sphere assuming that all vortices have equal strength. We develop a theoretical framework to analyse solutions of the equations of motion with prescribed symmetries. Our construction relies on the discrete reduction of the system by twisted subgroups of the full symmetry group that rotates and permutes the vortices. Our approach formalises and extends ideas outlined previously by Tokieda [3] and Soulière and Tokieda [2] and allows us to prove the existence of several 1-parameter families of periodic orbits. These families either emanate from equilibria or converge to collisions possessing a specific symmetry. Our results are applied to show existence of families of small nonlinear oscillations emanating from the Platonic solid equilibria. The talk is based on the preprint [1].

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On topological classification of dynamical systems with hyperbolic nonwandering sets

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According to S. Smale, a hyperbolic non-wandering set $NW(f)$ of a diffeomorphism $f : M^n \to M^n$ with dense set of periodic points in $NW(f)$ (M^n is a closed smooth manifold), is represented as a finite union of invariant closed sets, each of which contains a transitive orbit. These sets are called basic.

If the dimension of some basic set Λ of a diffeomorphism f is greater than one and coincides with the dimension of the supporting manifold M^n ($n > 1$), then Λ is unique and coincides with the whole manifold M^n. In this case, the diffeomorphism f is a diffeomorphism of D. Anosov. It follows from works of J. Franks and S. Newhouse, that if the dimensions of unstable or stable manifolds of points from Λ are equal to one, then the ambient manifold M^n is n-torus.

If the dimension of a basic set Λ of a diffeomorphism f is $n-1$, then Λ is either attractors or a repeller. The author of the report and E.V. Zhuzhoma have proved that if the non-wandering set of a structurally stable diffeomorphism f contains an orientable attractor (repeller) whose dimension is n-1 and coincides with the dimension of the unstable (stable) manifolds of its points, then the ambient manifold M^n is n-torus. Moreover, it was described recently by E. Zhuzhoma, V. Medvedev and the author of the report the topological structure of manifolds M^n ($n > 2$) admitting diffeomorphisms whose non wandering set consists of orientable expanding attractor and attracting repellers of dimension $n-1$.

If $n = 2$, then there is NW-stable diffeomorphism f on surface M^2 of any genus whose non-wandering set contains one-dimensional basic sets. Moreover, the topological structure of embedding them to ambient surface and dynamics of f are connected with the properties of basic sets and genus of ambient surface.

The report will be devoted to discussion of described results above and their application to the topological classification of structurally stable cascades on manifolds. To introduce with the topic of the report see books [1–3].

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Integrable maps on Stiefel manifolds

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We study integrable discretizations of geodesic flows of Euclidean metrics and of the Neumann systems on Stiefel manifolds $V_{n,r}$. In particular, starting from the discretization of the geodesic flow for $n = 3$, $r = 2$, after identifying $V_{3,2} \cong SO(3)$, we obtain a discrete analogue of the Euler case of rigid body motion corresponding to the inertia operator $I = (1, 1, 2)$. On the other hand, starting from the discrete Neumann system on $V_{3,3} = SO(3)$, but taking a different continuous limit, we have a well known Moser-Veselov discretization of the Euler top. In addition, billiard-type mappings are considered, one of them turns out to be the "square root" of the discrete Neumann system on $V_{n,r}$. The results are obtained jointly with Yuri Fedorov (UPC, Barselona).

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Experimental verification of the models of various robotic systems

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The paper presents the results of the experimental investigations of various mechanical and robotic systems. Due to the verification of the motion of simple systems, such as a rolling disk [1], a rolling ring [2], a spherical body with a displaced center of mass [3], it was possible to determine the limits of applicability of nonholonomic models of motion. A modification of nonholonomic models by introducing rolling resistance allows to implement them for description of spherical robots of different designs. Various modifications of spherical robots are described and their features and disadvantages are discussed: a spherical robot with internal rotors [4], a spherical robot with an internal omniwheel platform [5, 6], a spherical robot with a combined propulsion device [7–9].

The features of the practical implementation of models of mobile wheeled robots are also considered in the paper and algorithms for their control are proposed [10, 11].

In addition, the aquatic robots that implement screwless motion in a liquid are also considered. [12–14].

All discussed prototypes were fabricated in the Udmurt State University and in the laboratory of Mobile Systems of Kalashnikov Izhevsk State Technical University, and experimental studies were carried out.

The work was carried out within the framework of the state assignment of the Ministry of Education and Science of Russia (FZZN-2020-0011).

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Irregular solutions in the spatially distributed Fermi–Pasta–Ulam problem

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It is shown that various partial differential equations in the spatially distributed Fermi–Pasta–Ulam problem arise while describing the leading approximations of solutions in different domains of the phase space of the boundary value problem.

Special nonlinear boundary value problems are constructed to find the slowly varying amplitudes. Two cases differ. These boundary value problems are different for each of these two cases. In the first of them, the systems of two Schrödinger equations were obtained, in contrast to the second case where the system of two Korteweg – de Vries equations was obtained. Asymptotic representations of the irregular solutions contain superposition of functions depending on all of the following: the 'slow' time , the 'medium' time and 'fast' time. In addition, they contain periodic with respect to the spatial variables and rapidly oscillate components.

It follows formulas that the mutual influence of the functions ξ_+ and ξ_- leads only to the phase components change. If $\delta = 2\pi n_0$, then this influence is much weaker [1] when the higher infinitesimal order terms in the corresponding boundary value problems are taken into account. Therein, one can trace some analogies with the conclusions from [2–5].

A moving in one direction wave affects mainly only on the phase coordinate of a wave moving in the opposite direction. In this regard, a number of conclusions from the theory of solitons remain valid for irregular waves. We note also that the mutual effect of waves on each other differs significantly from the regular case.

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Hamiltonian geometry and the golden ratio in the Euler hydrodynamics

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The binormal (or vortex filament) equation provides the localized induction approximation of the 3D incompressible Euler equation. We present a Hamiltonian framework for the binormal equation in higher-dimensions and its explicit solutions that collapse in finite time. More generally, we also describe the geometry behind Newton's equations on infinite-dimensional configuration spaces of diffeomorphisms and smooth probability densities. On the other hand, by going to lower dimensions, we observe a curious appearance of the golden ratio in the motion of point vortices in the plane. This is a joint work with C.Yang and H.Wang.

Two problems of spherical bodies rolling on a vibrating plane

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Recently, the dynamics of rigid bodies on nonstationary surfaces, for example, rotating or vibrating ones, has attracted the attention of many researchers from all over the world. The motion of bodies on moving surfaces can be accompanied by various interesting dynamical phenomena. For example, in the case of a vibrating surface, the phenomenon of energy harvesting can be observed, which is described in the papers [1,2] concerned with rattleback dynamics. In this work we consider two problems of the rolling motion of spherical bodies on a vibrating plane [3,4].

The first problem is related to the motion of a Chaplygin sphere rolling without slipping on a plane performing horizontal periodic oscillations. For this problem we show that in the system under consideration the projections of the angular momentum onto the axes of the fixed coordinate system remain unchanged. The investigation of the reduced system on a fixed level set of first integrals reduces to analyzing a three-dimensional period advance map on $SO(3)$. The analysis of this map suggests that in the general case the problem considered is nonintegrable. We find partial solutions to the system which are a generalization of permanent rotations and correspond to nonuniform rotations about a body- and space-fixed axis. We also find a particular integrable case which, after time is rescaled, reduces to the classical Chaplygin sphere rolling problem on the zero level set of the area integral.

The second part of the work addresses the problem of a spherical robot having an axisymmetric pendulum drive and rolling without slipping on a vibrating plane. It is shown that this system admits partial solutions (steady rotations) for which the pendulum rotates about its vertical symmetry axis. Special attention is given to problems of stability and stabilization of these solutions. An analysis of the constraint reaction is performed, and parameter regions are identified in which a stabilization of the spherical robot is possible without it losing contact with the plane. It is shown that the partial solutions can be stabilized by varying the angular velocity of rotation of the pendulum about its symmetry axis, and that the rotation of the pendulum is a necessary condition for stabilization without the robot losing contact with the plane.

The work was carried out within the framework of the state assignment of the Ministry of Education and Science of Russia (FEWS-2020-0009).

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Some problems about the dynamics of vortex pairs on surfaces

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We suggest the idea that, like geodesics on a surface S with Riemannian metric g, the motion of a pair of opposite vortices could be useful for differential geometry in the large [1] — *only more so*.

While geodesic equations are local, vortex equations require the Green function $G(s_1, s_2)$ of the Laplace–Beltrami operator, thus reflecting the topology and geometry of the *whole* surface.

This is the case even for close by vortex pairs. For short times they shadow the geodesic flow with initial condition at the midpoint, as predicted by Y. Kimura in 1999 [2] and experimentally verified on the triaxial ellipsoid in [3]. But on a long time scale there is a slow drift proportional to to the square of the distance to the diagonal. Near the diagonal the energy levels are of contact type — thus containing periodic orbits, by Taubes' theorem for three dimensional contact structures.

We start by reviewing some result we published in RCD for genus 0 surfaces [4]. Not much is yet known about the vortex pair motion farther away from the diagonal. For genus 1 all the required machinery is available [5], but for genus ≥ 2 (automorphic surfaces) numerical experiments are in order to develop intutitions. Some results on Bolza's surface will be discussed [6]. Finally we suggest some problems for research, hoping to attract interest to the geometric analysis and to the symplectic geometry communities.

Some background information: Let S a Riemann surface with a specified metric g within a conformal class, with area form ω. We consider two opposite vortices moving in S. The symplectic form and Hamiltonian in $S \times S$ are given by

$$\Omega_{pair} = \pi_1 * \omega - \pi_2 * \omega \ , \quad (\pi_i \text{ the projections})$$
$$H = -G(s_1, s_2) + \frac{1}{2}\,(R(s_1) + R(s_2)\,.$$

Robin's function $R(s)$ is the regularization $R(s) = \lim_{s' \to s} G(s, s') - d(s, s')$. Near the diagonal it is useful to rescale the Hamiltonian as $F = \exp(-H)$, that can be written as

$$F(s_1, s_2) = d(s_1, s_2) \exp(B(s_1, s_2)) = d(s_1, s_2)(1 + O(d^2)),$$

where B, that we call Batman's function, is given by

$$B(s_1, s_2) = [G(s_1, s_2) - \ln d(s_1, s_2)] - \frac{1}{2}(R(s_1) + R(s_2)).$$

Two routes to study the dynamics near the diagonal are via the pullbacks of Ω_{pair} either by

$$E^{sympl} : v_s \in TS \to (s_, s_+) \in S \times S, \;\; s_\pm = \exp(\pm J v_s), \;\; J = \pi/2 \text{ rotation}$$
$$E^{folded} : (v_s, \alpha) \in T^1 S \times [-r, r] \to (s_-, s_+), s_\pm = \exp(s, \pm\alpha J v_s), \;\; |v_s| = 1.$$

In the former approach one introduces a scaling parameter, $s_\pm = \exp(\pm\epsilon J v_s)$ and one expands the pullback in powers of ϵ making the problem amenable to Hamiltonian perturbation theory.

In the latter, $2r$ is the injectivity radius, and we show that $M = T^1 S \times [-r, r]$ is a *folded symplectic space* and we describe its symplectic form via Jacobi fields along the geodesic joining s_-, s_+.

Near the diagonal

$$F = 2\alpha \exp\left(B(-\alpha, \alpha)\right) = 2\,\alpha\,(1 + m_2(v_s)\alpha^2 + O(\alpha^4)), \;\; v_s \in T^1 S$$

where $m_2 = Q(s)(v_s, v_s)$ is the quadratic term in the expansion of

$$B(s_-, s_+) = B(-\alpha, \alpha) = Q(s)(v_s, v_s)\,\alpha^2 + \cdots .$$

This function m_2 reflects the global influence in the perturbation of the geodesic system. In this sense a vortex dipole is a "topology sensor".

As an indication for this, we provide estimates on Green, Robin and Batman functions in terms of the injectivity radius r_S of the metric and first eigenvalue λ_1 of the hyperbolic Laplacian. Green functions transform nicely under conformal change of metrics. We think there are opportunities for research also for a Teichmüler theorist.

This is ongoing work with A.Regis, C. Castilho, C. Ragazzo, U. Hryniewicz, A. Cabrera and A.Aryasomayajula.

We dedicate this work to the memory of Alexey Borisov. We believe point vortices was one of his favorites themes, among many others of his interest. I counted fifteen papers on point vortices only in RCD.

Alexey visited Rio de Janeiro and Recife in 2011 and even before he was already a very good friend of Brazilian mathematics and Brazilian music. We are much indebted to him.

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Integrals of circulatory systems which are polynomial in momenta

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By circulatory systems one often means mechanical systems acted upon by nonpotential positional foces. The theory of circulatory systems is usually concerned with problems of the stability and bifurcations of equilibria and steady motions (taking into account additional gyroscopic and dissipative forces). This paper discusses the first steps in the theory of integration of circulatory systems. A treatment is given of a range of problems concerning conditions for the existence of first integrals, polynomial in momenta, with single-valued coefficients on configuration space. Topological obstructions to the existence of such integrals are found. Special attention is given to quadratic integrals. If the kinetic energy is “Euclidean”, then the existence of a nondegenerate quadratic integral allows the equations of motion to be reduced to Hamiltonian form. In the case of the “Liouvillian” kinetic energy, the equations are reduced to conformally Hamiltonian form. General results are illustrated by examples from the dynamics of circulatory systems.

Symplectic classification of structurally stable nondegenerate semilocal singularities

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Some of the results are obtained in a collaboration with A. Oshemkov.

An integrable Hamiltonian system on a symplectic $2n$-manifold (M,ω) is given by a smooth map $F=(f_1,\dots,f_n):M^{2n}\to\mathbf{R}^n$, called the *momentum map*, such that the functions f_i are functionally independent and pairwise in involution.

DEFINITION. Two integrable systems $F_i:M_i\to\mathbf{R}^n$, $i=1,2$, will be called *equivalent* (resp. *symplectically equivalent*) if there exist a homeomorphism (resp. symplectomorphism) $\Phi:M_1\to M_2$ and a homeomorphism $\phi:\mathbf{R}^n\to\mathbf{R}^n$ such that $\phi\circ F_1=F_2\circ\Phi$. The map Φ will be called an *equivalence* (resp. *symplectic equivalence*) between the systems.

A point $m\in M$ is called *singular* for F if $rank(dF(m))<n$.

By the Vey theorem [5], in real-analytic case, each nondegenerate singular point $m_0\in M$ has a neighbourhood in which the system is symplectically equivalent to a canonical system $(\mathbf{R}^{2n},\omega_{can},F_{can})$ whose momentum map $F_{can}=(h_1,\dots,h_n)$ is defined by *regular* components $h_j=x_j$ $(1\leqslant j\leqslant r)$, *elliptic* components $h_j=\frac{1}{2}(x_j^2+y_j^2)$ $(r+1\leqslant j\leqslant r+k_e)$, *hyperbolic* components $h_j=x_jy_j$ $(r+k_e+1\leqslant j\leqslant r+k_e+k_h)$ and *focus-focus* pairs of components $h_j=x_jy_j+x_{j+1}y_{j+1}$ and $h_{j+1}=x_{j+1}y_j-y_{j+1}x_j$ $(j=r+k_e+k_h+2i-1,\ 1\leqslant i\leqslant k_f)$, where $\omega_{can}=\sum\limits_{j=1}^{n}\mathrm{d}x_j\wedge\mathrm{d}y_j$. We say that the singular point m_0 has *Williamson type* (k_e,k_h,k_f) [6, Def. 2.3].

Notice that $r+k_e+k_h+2k_f=n$ and r is the rank of m_0. Clearly, the singular points of F in a small neighbourhood $U(m_0)$ of m_0 form $k_e+k_h+k_f$ symplectic submanifolds $\{dh_j=0\}\cap U(m_0)$, $j\in\{r+i\}_{i=1}^{k_e+k_h}\cup\{r+k_e+k_h+2i\}_{i=1}^{k_f}$ (called *critical submanifolds* of F on $U(m_0)$).

The momentum map F naturally gives rise to a (singular) Lagrangian fibration on M whose fibers are connected components of the level sets $F^{-1}(a)$, $a\in\mathbf{R}^n$. Consider the Hamiltonian $\mathbf{R}^n$-action on M generated by the momentum map $F=(f_1,\dots,f_n):M^{2n}\to\mathbf{R}^n$. We will call orbits of this action simply *orbits*. By a *local* (resp. *semilocal*) *singularity* of such a singular fibration, we mean the fibration germ at a singular orbit (resp. fiber).

Definition. We say that a non-degenerate singular fiber $L = F^{-1}(a)$ satisfies the *connectedness condition* if the number of connected critical submanifolds of F on a neighbourhood of L equals the number of critical submanifolds of F on a neighbourhood of each compact orbit $\mathcal{O} \subset L$.

Theorem 1 (Semilocal structural stability test [2]). *In real-analytic case, any compact non-degenerate singular fiber L satisfying the connectedness condition is structurally stable under real-analytic integrable perturbations (but not necessarily under C^∞ integrable perturbations) of the system. In other words, the integrable system (M, ω, F) on a neighbourhood $U(L)$ of the fiber L is equivalent to the perturbed system $(M, \tilde{\omega}, \tilde{F})$ on a slightly perturbed neighbourhood $\tilde{U}(L)$, for any (small enough) real-analytic integrable perturbation of the system.*

As an illustration, we obtain that a saddle-saddle singularity of the Kovalevskaya top is structurally stable under real-analytic integrable perturbations (but not under C^∞ perturbations).

Suppose that L_i, $i = 1, 2$, and L are compact non-degenerate singular fibres of real-analytic integrable systems (M_i, ω_i, F_i), $i = 1, 2$, and a reference system (M, ω, F), resp. Suppose the singularities at L_i are equivalent to the reference singularity at L, and $\Phi_i : U(L) \to U(L_i)$ is an equivalence, with $\Phi_i(L) = L_i$. Consider the Hamiltonian $(S^1)^{r+k_e+k_f}$-action near L generated by the corresponding components h_j, $j \in T := \{1, \ldots, r + k_e\} \cup \{r + k_e + k_h + 2i\}_{i=1}^{k_f}$, of the canonical momentum map $F_{can} = (h_1, \ldots, h_n)$ at a compact orbit $\mathcal{O} \subset L$. Consider the *reduced fiber* $X = L/(S^1)^{r+k_e+k_f}$.

Suppose that Φ_i is a real-analytic symplectomorphism near the orbit $\mathcal{O}$, $i = 1, 2$. We say that a symplectic equivalence $\Phi : U(L_1) \to U(L_2)$ is *good* if its restriction to a small neighbourhood $U(\mathcal{O}_1)$ of the orbit $\mathcal{O}_1 = \Phi_1(\mathcal{O}) \subset L_1$ coincides with the symplectomorphism $\Phi_2 \circ \Phi_1^{-1}|_{U(\mathcal{O}_1)}$.

If L_1 satisfies the connectedness condition, we introduce the *Lagrange-Vey class* of the singularity at L_1 (w.r.t. the given reference singularity at L), which is a 1-cocycle

$$\beta_{L_1} \in H^1(X, Z^1/R) \cong Hom(H_1(X), Z^1/R). \qquad (*)$$

Here $(Z^1, +)$ is the group of converging (near the origin) power series in n variables with real coefficients, vanishing at the origin, and $R \subset Z^1$ is the subgroup of rank $k_e + k_f$ generated by the monomials z_j, $j \in T$.

Theorem 2. *Under the hypothesis from above, the singularities at L_1 and L_2 are symplectically equivalent via a good symplectic equivalence if* *and only if these singularities have the same Lagrange-Vey class* $(*)$*, i.e.* $\beta_{L_1} = \beta_{L_2}$*. For each 1-cocycle* $[\beta] \in H^1(X, Z^1/R)$*, there exists a semilocal singularity* L_1 *equivalent to* L*, whose Lagrange-Vey class is* $[\beta]$.

Note that Theorem 2 extends (the real-analytic versions of) the semilocal symplectic classifications of simple Morse functions on compact symplectic surfaces [1] and semitoric systems [3,4].

Theorem 3. *Under the hypothesis from above, suppose that there exists a finite-sheeted covering of a neighbourhood of* L *that is fibrewise homeomorphic to the direct product of several regular, elliptic, hyperbolic and focus-focus semilocal singularities of dimensions* 2, 2, 2 *and* 4*, resp. Suppose that all hyperbolic components (2-atoms) of this decomposition have genus 0. Then the singularities at* L_1 *and* L_2 *are symplectically equivalent via a good symplectic equivalence if and only if the map* $\Phi_2 \circ \Phi_1^{-1}$ *preserves the action variables.*

Thus, the action variables form a complete set of semilocal symplectic invariants.

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Existence of liouvillian solutions in the problem of motion of a heavy rigid body with a fixed point in the Hess case

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In 1890 German mathematician and physicist W. Hess [1] found new special case of integrability of Euler – Poisson equations of motion of a heavy rigid body with a fixed point. In 1892 P. A. Nekrasov proved [2] that the solution of the problem of motion of a heavy rigid body with a fixed point under Hess conditions reduces to integrating the second order linear differential equation. To obtain this equation we firstly derive the Euler – Poisson equations in the special Kharlamov coordinate system [3, 4]. Using this form of the Euler – Poisson equations we derive the corresponding linear differential equation and transform its coefficients to the form of rational functions. Applying the Kovacic algorithm [5] to the obtained differential equation, we proved that the liouvillian solutions of the corresponding second order linear differential equation exists only in the case, when the moving rigid body is the Lagrange top, or in the case, when the constant of the area integral is zero [6].

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Resonances in the stability problems of a Thomson vortex N-gon inside/outside circular domain and a point vortex quadrupole on a plane

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A review of studies of resonant cases in the stability problem of a Thomson vortex N-gon inside and outside a circular domain is given [1, 2]. It is noted that two of them lead to instability: a double-zero resonance in the case $N = 3$ and a $1 : 2$ resonance in the case $N = 5$. The known results of the stability theory of the resonant cases were used [3].

In addition, a system of four point vortices on a plane is considered. Its motion is described by the Kirchhoff equations. Three vortices have unit intensity and one vortex has arbitrary intensity κ. We study the stability problem for the stationary rotation of a point vortex quadrupole consisting of three identical vortices located uniformly on a circle around a fourth vortex. It is known that for $\kappa > 1$ the investigated regime is unstable [4] (the linearized system has exponentially growing solutions). In the case of $\kappa < -3$ and $0 < \kappa < 1$ the orbital stability takes place. New results are obtained for $-3 < \kappa < 0$, [5]. It was found that for all values κ in the stability problem there are a resonance $1 : 1$ (diagonalizable case). Some other resonances up to the fourth order inclusive are found and investigated: double zero resonance (diagonalizable case), resonances 1:2 and 1:3, occurring at isolated values κ. The stability of the equilibrium of the system reduced by one degree of freedom with the involvement of the terms in the Hamiltonian up to the fourth degree inclusive is proved for all $\kappa \in (-3, 0)$.

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Topological aspects of stability

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I will describe two observations on the motion of coupled particles in periodic potentials. Coupled pendula, or the space-discretized sine-Gordon equation is an example of this problem. Linearized spectrum of the synchronous motion turns out to have a hidden asymptotic periodicity in its dependence on the energy; this is the gist of the first observation. Our second observation is to point out a special property of the purely sinusoidal potentials: the linearization around the synchronous solution is equivalent to the classical Lamè equation. As a consequence, it turns out that *all but one instability zones of the linearized equation collapse to a point for the one-harmonic potentials*. This seems to be a new example where Lamé's finite zone potential arises, and in the simplest possible setting. This also shows that the higher harmonics contribute to instability. The latter phenomenon bears some analogy to the loss of sharpness of Arnold tongues in circle maps in the presence of higher harmonics, and also to the loss of sharpness of instability zones in Mathieu equation when higher harmonics are present in the potential. This is joint work with J. Zhou and K. Kim.

Integrability of Hamiltonian systems with gyroscopic term

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We consider systems with two degrees of freedom for which Hamilton has the form

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \omega(p_1 q_2 - p_2 q_1) + V(q_1, q_2). \tag{1}$$

The therm proportional to ω is called gyroscopic. The notion of the velocity dependent potential is also used. Our aim is to investigate the integrability of such systems. Some investigations concerning this problem have been done. However, mostly they were restricted to searching additional first integrals which are linear or quadratic in momenta, see for example [1–6].

Our main result is the following theorem.

Theorem 1. *System given by Hamiltonian (1), with $\omega \neq 0$ and rational homogeneous l potential $V(q_1, q_2) \in \mathbb{C}(q_1, q_2)$ of degree $k \in \mathbb{Z}$, $|k| > 2$, such that $V(1, \mathrm{i}) \neq 0$ does not admit any rational first integral which is functionally independent with H.*

We prove this theorem using methods of differential Galois theory. Possible generalisations of this result for non-homogeneous potential and more general forms of the gyroscopic terms are discussed.

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Dynamics of an unbalanced circular foil and point vortices in an ideal fluid

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We consider the problem of the motion of an unbalanced circular foil and point vortices in an ideal incompressible fluid. Using Bernoulli's theorem for unsteady potential flow, the force due to the pressure from the fluid on the foil is obtained for an arbitrary vortex motion. The pressure force for a circular foil undergoing motion in the presence of one vortex was first obtained in [1]. In [2,4], the equations of motion for a foil were obtained in a fixed coordinate system, and those for a point vortex, in a coordinate system attached to the foil. In these variables, the Poisson structure and the additional integrals are given by fairly cumbersome expressions. The equations of motion for a foil and vortices in the same fixed (inertial) coordinate system were derived anew in [3]. This made it possible to represent in a natural way the equations of motion in Hamiltonian form with a canonical Poisson bracket and to find first integrals due to the symmetry of the system.

A detailed analysis is made of the case of free vortex motion in which a Hamiltonian reduction by symmetries is performed. For the resulting system, relative equilibria corresponding to the motion of an unbalanced foil and a vortex in a circle or in a straight line are found and their stability is investigated. New examples of stationary configurations of a vortex and a foil are given. Using a Poincare map, it is also shown that in the general case of an unbalanced circular foil the reduced system exhibits chaotic trajectories. In addition, a detailed qualitative analysis of the dynamics of the integrable case of a balanced foil and a vortex is carried out.

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On the identical resonance and stability of Lagrangian solutions to the bounded three-body problem

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The bounded problem of three bodies (material points) is considered. It is assumed that the orbits of the main attracting bodies are ellipses of small eccentricity, and that, as the passively gravitating body moves, it may leave the plane of the orbits of the main bodies (which amounts to the spatial problem). An analysis is made of the stability of the motion of this body, which corresponds to triangular Lagrangian libration points. A characteristic feature of the spatial problem under study is the existence of a resonance due to equality between the period of the Keplerian motion of the main bodies and the period of linear oscillations of the passively gravitating body in the direction perpendicular to the plane of their orbits (identical resonance). Using the methods of classical perturbation theory, KAM theory and algorithms of computer algebra, a treatment is given of the nonlinear problem of stability for most (in the sense of Lebesgue measure) initial conditions and of the problem of formal stability (stability in an arbitrarily high finite approximation with respect to the coordinates and momenta of the perturbed motion).

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Commuting differential and difference operators

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We will discuss the connection between commuting differential operators and commuting difference operators.

On the phase change for perturbations of one-frequency systems with separatrix crossing

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We study the evolution of angular variable (phase) for general (not necessarily Hamiltonian) perturbations of Hamiltonian systems with one degree of freedom near the separatrices of the unperturbed system. To this end we use averaged system of order 2. We obtain estimates of the accuracy of averaged system of order 2 near separatrices and use these estimates to prove a formula for the phase change when solutions of the perturbed system approach the separatrices of the unperturbed system (such formula is known when the perturbation is Hamiltonian). As an application of this formula, we show that two natural definitions of probability of capture into different domains after separatrix crossing proposed by V. I. Arnold and D. V. Anosov lead to the same formula for this probability.

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Apsidal alignment in double averaged restricted elliptic three-body problem: stability analysis

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We are dealing with the averaged model used to study the secular effects in the motion of a body of the negligible mass in the context of a spatial restricted elliptic three-body problem. It is supposed that the mass of one of the primaries is significantly greater than the mass of the other (a more massive primary body will be called a star, a less massive one will be called a planet). The double averaged restricted elliptic three-body problem admits a two-parameter family of equilibria (stationary solutions) corresponding to the motion of the third body in the plane of primaries' motion, so that the apse line of the orbit of this body is aligned with the apse lines of the primaries' orbits [1, 2]. The aim of our investigation is to analyze the stability of these alignments.

We start by proving the stability of the stationary apsidal alignment in the linear approximation. Then Arnold – Moser stability theorem [3, 4] is applied to obtain the sufficient conditions under which the stability in a nonlinear sense takes place. These conditions are satisfied for all parameters of the problem, with the exception of parameters from some finite set of analytic curves in space of parameters. The exceptional values of parameters correspond to resonances 1:1 and 2:1 between frequencies of oscillations of the apse line in the plane of primaries' motion and across this plane and to a degeneration of 4th order Birkhoff normal form of the problem's Hamiltonian.

Assuming that the semi-major axis of the orbit of the body of negligible mass is significantly greater or significantly less than the semi-major axis of the planet's orbit, we investigated what happens when the conditions of the Arnold – Moser theorem are violated. It turned out that in the case of 2:1 resonance apsidal alignment is unstable. In other cases, the violation of the conditions of Arnold – Moser theorem does not lead to instability.

Our results hopefully will be useful for studying the dynamics of exoplanetary systems. It is believed that apsidal alignments do occur in some of them (e.g., [5]).

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The influence of the friction model on the inversion of the tippe top

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In this work, we analyze the effect which the choice of a friction model has on tippe top inversion in the case where the resulting action of all dissipative forces is described not only by the force applied at the contact point, but also by the additional rolling resistance torque [1]. It turns out that depending on the chosen friction model, the system admits different first integrals: the Jellett integral, the Lagrange integral or the area integral.

In the classical model explaining the top inversion [2, 3], the rolling resistance is described only by the friction force applied at the point of contact. As is well-known, in this case, the Jellett integral is preserved and top inversion is possible regardless of the specific type of friction force satisfying the condition of energy dissipation. The proof of top inversion is based on the stability analysis of steady rotations depending on the system parameters and the value of the first integral of motion. As shown in [2, 3], the existence and stability of steady rotations is completely determined by the integral of motion and does not depend on the specific type of friction force.

In our work we show that the possibility or impossibility of tippe top inversion depends on the existence of specific integrals of the motion of the system. We consider an example of the law of rolling resistance by which the area integral is preserved in the system. We examine in detail the case where the action of all dissipative forces reduces to the horizontal rolling resistance torque. This model describes fast rotations of the top between two horizontal smooth planes. For this case, we find permanent rotations of the system and analyze their linear stability. The stability analysis suggests that no tippe top inversion is possible under fast rotations between two planes. Following [2, 3], one can show that, when the area integral is preserved, top inversion is impossible for any law of rolling resistance satisfying the condition of energy dissipation. An explicit proof of this fact can be the subject of a separate study.

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3-Diffeomorphisms with dynamics “one-dimensional surfaced attractor-repeller”

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The dynamics of any Ω-stable diffeomorphism f of a closed 3-manifold M^3 can be represented as an attractor-repeller. If a diifeomorphism $f : M^3 \to M^3$ has a one-dimensional hyperbolic attractor and repeller then the attractor (the repeller) is automatically expanding (contracting) as it consists of the unstable (stable) manifolds of its points, that was proved by R. Plykin [5]. R. Williams [7] shows that the dynamics on such a basic set is conjugate to the shift on the reverse limit of a branched 1-manifold with respect to an expanding map. A construction of 3-diffeomorphisms with one-dimensional attractor-repeller dynamics firstly was suggested by J. Gibbons [3]. He construct many models on 3-sphere with Smale’s solenoid basic sets and proves that all examples are not structurally stable. B. Jiang, Y. Ni and S. Wang [4] proved that a 3-manifold M^3 admits a diffeomorphism f whose non-wandering set consists of Smale’s solenoid attractors and repellers if and only if M^3 is a lens space $L(p, q)$ with $p \neq 0$. They also shown that such f are not structural stable.

All generalizations of Smale’s solenoid as the intersections of nested handlebodies are not surface. Moreover, all known examples of diffeomorphisms with the generalized solenoids as the attractor and the repeller are not structurally stable. A natural way to get a surface one-dimensional attractor for a 3-diffeomorphism f is to take an attractor A of some 2-diffeomorphism and multiply its trapping neithborhood by a contraction in transversal direction. According to [1] such attractor A is called *canonically embedded surface attractor*. One says that R is a *canonically embedded surface repeller* of f if it is a surface attractor for f^{-1}.

Infinitely many pairwise Ω-non-conjugated diffeomorphisms with such attractors and repellers were constructed in [1]. Moreover, there a conjecture was formulated that all such diffeomorphisms are not structurally stable. The main result of this paper is the proof of the conjecture.

Theorem. *There are no structurally stable 3-diffeomorphisms whose non-wandering set is a disjoint union of one-dimensional hyperbolic canonically embedded surface attractor and repeller.*

Notice, that in [2], [6] structurally stable 3-diffeomorphisms with one-dimensional attractor-repeller dynamics were constructed, but the constructed basic sets were not canonically embedded in surfaces in that examples.

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The Kapitza – Whitney pendulum

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A generalization of the classical Kapitza pendulum is considered: an inverted planar mathematical pendulum with a vertically vibrating pivot point in a time-periodic horizontal force field. The dynamics of this system is more complex than of the classical Kapitza pendulum. However, It has been previously shown that there always exists a periodic solution along which the rod of the pendulum never becomes horizontal, i. e., the pendulum never falls, provided the period of vibration and the period of horizontal force are commensurable. We present a numerical study of stability of these non-falling periodic solutions.

Non-integrability of planar elliptic restricted three body problems

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We analyse the integrability of various planar elliptic restricted three body problems: the classical problem, its photogravitational generalization and the planar elliptic limiting Hill's problem. In the classical restricted three body problem we consider the dynamics of a point with a negligible mass moving in the gravity field of two other points masses called the primaries with huge masses m_1 and m_2. They move in elliptic Keplerian orbits around their common mass centre. In the case when massless test particle moves in the same plane as the orbits of the primaries the problem is called planar elliptical restricted three body problem, otherwise is called spatial one. The mass parameter is $\mu = \frac{m_2}{m_1+m_2}$, $m_2 \leqslant m_1$, $\mu \in (0, 1/2]$. We assume that $m_1 + m_2 = 1$ and then primaries have masses: heavier $m_1 = 1 - \mu$ and lighter one $m_2 = \mu$, respectively.

The planar elliptic restricted three-body problem possesses the generalisation which takes into account the radiation of the primaries. It is called the photogravitational elliptic restricted three body problem.

To describe dynamics of the massless particle we pass to the frame which rotates nonuniformly and pulsates isotropically in the plane to ensure that the primary masses remain fixed at the positions $P_1 = (-\mu, 0)$ and $P_2 = (1 - \mu, 0)$. If (q_1, q_2) denotes the position of the massless particle in this frame, then its equations of motion are determined by Hamiltonian

$$\begin{aligned} H_{\mathrm{r3bp}} = \frac{1}{2}(p_1^2 + p_2^2) + p_1 y - p_2 x + \frac{er(e,\nu)\cos\nu}{2}(q_1^2 + q_2^2) - \\ - r(e,\nu)\left(\frac{\sigma_1(1-\mu)}{r_1} + \frac{\sigma_2\mu}{r_2}\right), \qquad (1) \\ r_1 = \sqrt{(q_1+\mu)^2 + q_2^2}, \quad r_2 = \sqrt{(q_1 - 1 + \mu)^2 + q_2^2} \end{aligned}$$

where $e \in (0, 1)$ is the eccentricity and function $r(e, \nu)$ equals

$$r(e,\nu) = \frac{1}{1 + e\cos\nu}. \qquad (2)$$

Here $\nu(t)$ is the true anomaly and constants $\sigma_i = 1 - \beta_i$, $i = 1, 2$ measure strength of the radiation-pressure forces. More precisely, $\beta = F_{\mathrm{r}}/F_{\mathrm{g}}$, where

F_{r} is the radiation-pressure force and F_{g} is the gravitational force of the respective primary. Constants $\sigma_i \in [0, 1]$, where $\sigma_i = 1$ corresponds the previously described classical case without radiation while $\sigma_i = 0$ corresponds situation when radiation-pressure force balances the gravitational force. The restricted three body is very popular and still very actively considered problem of celestial mechanics, see e.g. [2, 4, 8].

In the limiting Hill case of the restricted three body problem the massless body is attracted by two primary bodies one of which is at extremely larger distance from the second one and we are interested in motion of massless body near the smaller primary mass. In order to pass to this limit we make transformation $(q_1, q_2, p_1, p_2) \to (Q_1, Q_2, P_1, P_2)$

$$q_1 = Q_1\mu^{1/3}+1-\mu, \quad q_2 = Q_2\mu^{1/3}, \quad p_1 = P_1\mu^{1/3}, \quad p_2 = P_2\mu^{1/3}+1-\mu.$$

This transformation moves the smaller primary with mass $\mu < \frac{1}{2}$ placed to the right at $(1-\mu, 0)$ to the origin and applied rescaling in the limit $\mu = 0$ sends the more massive primary to infinite distance. Also in this limit it is necessary to change the radiation coefficient of the larger primary, thus we make transformation $(\sigma_1, \sigma_2) \to (s_1, s_2)$ given by $\sigma_1 = 1 - s_1\mu^{1/3}$ that means $\beta_1 = s_1\mu^{1/3}$ and $\sigma_2 = s_2$.

The dominant term in the expansion of Hamiltonian (1) in powers of $\mu^{1/3}$ gives the so-called Hill limit

$$\begin{aligned} H_{\mathrm{Hill}} = &\frac{1}{2}(P_1^2 + P_2^2) + P_1Q_2 - P_2Q_1 - s_1r(e,\nu)Q_1+ \\ &+ \frac{1}{2}\left[Q_1^2\left(1 - 3r(e,\nu)\right) + Q_2^2\right] - \frac{s_2r(e,\nu)}{\sqrt{Q_1^2+Q_2^2}}, \end{aligned} \tag{3}$$

for details see e.g. [7].

Hamiltonian equations generated by (1) or by (3) have equilibria called the Lagrange points and we will consider dynamics in a neighbourhood of the so-called triangular Lagrange points. The natural application of these variational equations is the linear stability analysis of the triangular points that subject became popular in the 1960s [3, 5, 9].

But there is also other application of these variational equations. Namely, because these systems are time-periodic Hamiltonian with two degrees of freedom variational equations around these equilibria are linear equations with non-constant coefficients. Using these equations we show non-integrability by means of the differential Galois integrability obstructions [6]. We consider all variational equations and the so-called symplectic Kovacic's algorithm [1]. Variational equations rewritten as the fourth order scalar equations factorize into two second order differential operators. Analysing differential Galois group of the right-hand factor enables to show the non-integrability.

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On asymptotics of Painlevé transcendents

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Using the construction of the initial values space for Painlevé differential equations, we analyse the asymptotics of their solutions around the singularities. The results are obtained jointly with Nalini Joshi.

Quantum phase space localization in chaotic systems

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Dedicated to the memory of Alexey Borisov

Quantum localization (or dynamical localization) in systems that are classically chaotic (either ergodic and fully chaotic or partially chaotic, of the mixed-type) is one of the central phenomena in quantum chaos. The phase space structure of the chaotic eigenstates is studied by means of Wigner functions or Husimi functions. It turns out that they are ergodic (maximally extended in the chaotic part of the phase space) if the Heisenberg time scale of the system with the discrete energy spectrum is larger than the classical transport time scale (for the diffusion in the momentum space), whilst if this condition is not met the chaotic eigenstates are localized in the chaotic part of the phase space. We can quantify the degree of localization by various localization measures, such as the entropy localization measure, the correlation localization measure, or the normalized inverse participation ratio of the Husimi functions. They are all linearly related and thus equivalent. The transition from localized to completely delocalized (ergodic) regime is rather smooth. Also, the spectral statistics is affected strongly by the quantum localization. For example, in fully chaotic systems the level spacings distribution is well described by the Brody distribution, while in the case of the mixed-type systems it is BRB (Berry–Robnik–Brody). The Brody level repulsion parameter β goes from 0 in case of the strongest localization (implying Poisson distribution) to 1 in case of complete extendedness on the chaotic component (implying the GOE distribution well approximated by the Wigner distribution for the chaotic part of the energy spectrum). In the extreme special case of ergodic regime with full chaos (no regular component) and no quantum localization the statistics is well described by the Gaussian random matrix theories (GOE). I shall explain the general theoretical approach and present the results for the billiard systems as well as for the Dicke system. The latter one is a mixed-type system with a classical analogue having a smooth potential.

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Sub-Riemannian geometry on the group of motions of the plane

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We will discuss the unique, up to local isometries, contact sub-Riemannian structure on the group $\mathrm{SE}(2)$ of proper motions of the plane (also known as the group of rototranslations).

The following questions will be addressed:

- geodesics,
- their local and global optimality,
- cut time, cut locus, and spheres,
- infinite geodesics,
- bicycle transform and relation of geodesics with Euler elasticae,
- group of isometries and homogeneous geodesics,
- applications to imaging and robotics.

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Steady states of the Vlasov equation with modified potential of the Lennard – Jones type

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We consider the gravitating particles that can collide. Collisions can be described in various ways. We can use the theory of inelastic interaction of solids with Newton's recovery coefficient for the relative velocity of colliding particles. In numerical implementation, the main difficulty of this approach is to track and refine a huge number of time moments of particle collisions. Another approach is to add to the gravitational potential the potential of repulsive forces, similar to the intermolecular Lennard – Jones forces. Numerical experiments show that when the Jacobi stability condition is satisfied, both models lead to a qualitatively identical character of evolution with the possible formation of stable configurations [1]. As it is known, when pair collisions of an infinitely large number of gravitating particles are taken into account, the probability density function evolves in accordance with the Vlasov – Boltzmann – Poisson system of equations [2]. We suggest a research method using the Vlasov equation with the Lennard – Jones type potential. This allows to take into account the size of the interacting particles, and also take into account not only paired, but also triple or more collisions of the particles. For this dynamical system the existence of a large class of nonlinearly stable equilibrium solutions is proved by the Energy – Casimir method [2].

The main point of this study is a general method for asserting nonlinear stability for infinite-dimensional Hamiltonian systems. Our strategy for proving the existence of nonlinearly stable stationary states will be as follows. The phase flow of the equations of characteristics corresponding to the Vlasov equation for the problem with a modified gravitational potential preserves the phase volume. Then, for any reasonably chosen function, the so-called Casimir functional — the integral of this function of the phase density over the phase space — will also be preserved. The Hamiltonian, which is a functional of the phase density, does not have critical points if we take the space of all densities of the phase space as the state space. The linear part of the continuation in the vicinity of some stable state with the corresponding potential does not disappear, but for the energy — Casimir functional (H + C) the corresponding steady states are critical points [3, 4]. (Critical points of the Hamiltonian restricted to the manifold, which is determined by the preservation of the Casimir functional).

The paper proves the existence of Lyapunov stable equilibrium solutions of the Vlasov equation describing the evolution of the phase density function of a system of mutually gravitating particles with possible collisions. Instead of analyzing the stability of a particular equilibrium state, the functional of the phase density function is investigated – whether this functional will reach a minimum on a suitable set of states f. Such minimizer, if it exists, is a critical point of the energy functional, and its minimization property leads to the statement of dynamic stability. The key point of this paper is the proof of the theorem on the existence of a minimizer of the energy functional on the bounding set defined by the given constants [5].

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Reflection and refraction of Lagrangian manifolds and Maslov complex germs corresponding to short-wave solutions of the wave equation with an abruptly varying velocity

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We describe the propagation of waves in media containing localized rapidly changing inhomogeneities (for example, narrow underwater ridges or pycnoclines in ocean, layers with drastically changing optical or acoustic density, etc.). We study behavior of Lagrangian surfaces and Maslov complex germs, corresponding to short-wave solutions; in particular, we describe refraction and reflection of these geometrical objects on the support of the inhomogeneity.

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New stationary states of three point vortices in a two-layer rotating fluid

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The integrable two-dimensional problem of three vortices [1, 2] has attracted the interest of researchers for over 125 years [5]. This is not associated only with the vortex problem in itself but also has numerous analogies in the mechanics of solids, astrophysics, the dynamics of superfluid helium, and mathematical biology [1]. A new peak of interest in this problem was stimulated by the discovery of so-called three-polar structures [4], i.e., symmetric triples of vortices $(-\kappa, 2\kappa, -\kappa)$ and by later observation of their spontaneous origin from chaos [3]. In most studies, the dynamics of vortices was analyzed in the framework of the homogeneous-fluid model. At the same time, the geophysical problems are characterized by noticeable density stratification and rotation [6].

Here, we suppose that the vortex motion takes place in an unbounded rotating two-dimensional two-layer fluid with constant densities ρ_1, ρ_2 ($\rho_1 < \rho_2$) in the upper and lower layer, respectively. Let one of the vortices be situated in the upper layer and two vortices in the lower layer. The equations of motion in the coordinate system x; y rotating along the axis perpendicular to plane (x, y) with an angular velocity of Ω take the form [6]

$$
\begin{aligned}
\dot{x}_1^1 &= -\frac{1}{4\pi}\sum_{\alpha=1}^{2}\kappa_2^\alpha \frac{y_1^1 - y_2^\alpha}{\left(r_{12}^{1\alpha}\right)^2}\left[1 - \gamma r_{12}^{1\alpha}\mathrm{K}_1\left(\gamma r_{12}^{1\alpha}\right)\right],\\
\dot{x}_2^\alpha &= -\frac{1}{4\pi}\left\{\kappa_2^{3-\alpha}\frac{y_2^\alpha - y_2^{3-\alpha}}{\left(r_{22}^{\alpha(3-\alpha)}\right)^2}\left[1 + \gamma r_{22}^{\alpha(3-\alpha)}\mathrm{K}_1\left(\gamma r_{22}^{\alpha(3-\alpha)}\right)\right]\right.\\
&\qquad \left. + \kappa_1^1\frac{y_2^\alpha - y_1^1}{\left(r_{21}^{\alpha 1}\right)^2}\left[1 - \gamma r_{21}^{\alpha 1}\mathrm{K}_1\left(\gamma r_{21}^{\alpha 1}\right)\right]\right\}, \qquad (1)\\
\dot{y}_1^1 &= \frac{1}{4\pi}\sum_{\alpha=1}^{2}\kappa_2^\alpha \frac{x_1^1 - x_2^\alpha}{\left(r_{12}^{1\alpha}\right)^2}\left[1 - \gamma r_{12}^{1\alpha}\mathrm{K}_1\left(\gamma r_{12}^{1\alpha}\right)\right],\\
\dot{y}_2^\alpha &= \frac{1}{4\pi}\left\{\kappa_2^{3-\alpha}\frac{x_2^\alpha - x_2^{3-\alpha}}{\left(r_{22}^{\alpha(3-\alpha)}\right)^2}\left[1 + \gamma r_{22}^{\alpha(3-\alpha)}\mathrm{K}_1\left(\gamma r_{22}^{\alpha(3-\alpha)}\right)\right]\right.
\end{aligned}
$$

$$+\kappa_1^1 \frac{x_2^\alpha - x_1^1}{\left(r_{21}^{\alpha 1}\right)^2}\left[1-\gamma r_{21}^{\alpha 1} \mathrm{K}_1\left(\gamma r_{21}^{\alpha 1}\right)\right]\Bigg\}. \tag{2}$$

Here, κ_j^α is the intensity of the point vortex with the coordinates (x_j^α, y_j^α). The subscript and the superscript correspond to the layer number and to vortex number, respectively (in our case, $\alpha = 1$ at $j = 1$, and $\alpha = 1, 2$ at $j = 2$); $r_{ij}^{\alpha\beta} = \sqrt{(x_j^\beta - x_i^\alpha)^2 + (y_j^\beta - y_i^\alpha)^2}$; γ is inversely proportional to the Rossby internal deformation radius $\lambda = \sqrt{g' h_1 h_2/(h_1 + h_2)}/2\Omega$, g' the reduced acceleration of gravity, h_1 and h_2 are the thickness of the upper and lower layers, respectively. In (1)–(2) and bellow we assume $h_1 = h_2 = 1/2$; K_1 is the modified Bessel Function of the first order.

One of the main integral invariants $L = \kappa_2^2\kappa_2^1 r_{22}^{21} + \kappa_1^1\kappa_2^2 r_{12}^{12} + \kappa_1^1\kappa_2^1 r_{12}^{11}$ is of fundamental importance for the motions of a vortex structure. The motions of such a vortex structure are very diverse. At $\kappa = \kappa_1^1 + \kappa_2^1 + \kappa_2^2 \neq 0$, the trajectories of all vortices are finite and, with time, completely obscure the concentric annular regions centered at the point $(x_c, y_c) = ((\kappa_1^1 x_1^1 + \kappa_2^1 x_2^1 + \kappa_2^2 x_2^2)/\kappa, (\kappa_1^1 y_1^1 + \kappa_2^1 y_2^1 + \kappa_2^2 y_2^2)/\kappa)$. Among these solutions, one can find

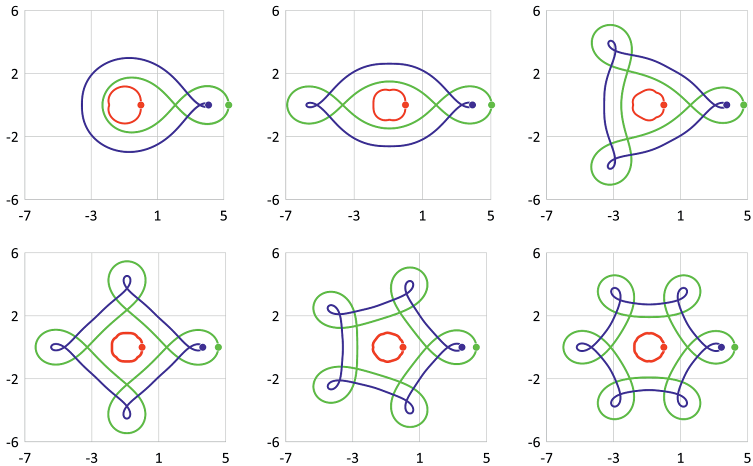

Fig. 1. The family of choreography for vortices of the upper (red line) and lower (green and blue lines) layers for stationary solutions of the m-modal type ($m = 1, 2, \ldots, 6$). Circle markers indicate the initial positions of the vortices. Here, L=6; $\kappa_1^1 = 4$, $\kappa_2^1 = 2$, $\kappa_2^2 = -1$; and in initial time $x_1^1 = y_1^1 = y_2^1 = y_2^2 = 0$; x_2^2= 4.0824, 3.9242, 3.7604, 3.6287, 3.5242, 3.4400 for m from 1 to 6, respectively. The x_2^1 values are calculated in terms of x_2^2 and L

a countable number of stationary periodic solutions with closed trajectories (so-called choreographies) of m-modal type (the mode number coincides with the number of peripheral vortex loops of the lower layer. Examples of such trajectories are shown in Fig. 1 for the case when at the initial moment of time all vortices are located on the x-axis.

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Cusps of caustics by reflection and Legendrian knots

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Assume that the boundary of a planar oval is an ideal mirror, and one has a point source of light inside the oval. Consider the rays of light that have undergone N reflection in the mirror, where $N = 1, 2, \ldots$ The envelope of this family of rays is the Nth caustic by reflection. I shall show that, for every N, this caustic has at least four cusps. This result is a consequence of a far reaching generalization of the 4-vertex theorem, conjectured by Arnold and proved by Chekanov and Pushkar' using the Legendrian knot theory [1].

Similar problems for convex surfaces were considered before: Caratheodory proved that the locus of points conjugated to a given point has at least four cusps, and Jacobi stated, in his "Lectures on dynamics", that this number is exactly four in the case of the ellipsoids (this is known as the "Last Geometric Statement of Jacobi"). Our problem is a billiard version of these problems of differential geometry of surfaces. Conjecturally, all caustics by reflection in ellipses have exactly four cusps, and this property is characteristic for ellipses.

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Exact solutions of the Davey – Stewartson equation and minimal surfaces in the four-space

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We construct the Moutard type transformation for solutions of the Davey – Stewartson equation. By this transformation and the spinor (Weierstrass) representation of surfaces in the four-space, we construct from the minimal surfaces in $\mathbb{C}^2 = \mathbb{R}^4$ of the form $z_2 = f(z_1)$ where $(z_1, z_2) \in \mathbb{C}^2$ and f is a holomorphic function exact solutions of this equation such that they have regular and fast decaying inital data and lose regularity in a final time.

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Dynamics of a class of simple mobile parametric oscillators

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Dedicated to the memory of Alexey Borisov

Terrestrial locomotion that is produced by creating and exploiting frictional anisotropy is common amongst animals such as snakes, gastropods, limbless lizards. This has inspired research in designing robots and mechanisms with substrates that possess frictional anisotropy. This talk will describe a model of a bristle bot,a platform that rests on a substrate supported by bristles, that locomotes by generating frictional anisotropy due to the oscillatory motion of an internal mass. Such vibrational robots have been available as toys and theoretical curiosities and have seen some applications as the well known kilobot and in pipe line inspection, but much remains unknown about such type of terrestrial locomotion. The principle of vibrations induced motion has been studied in idealized mechanical systems moving on surfaces with anisotropic friction in [1–3]. A quasi static analysis of the bristle bot dynamics in [4] showed the friction between the bristle tips and the ground are different in different phases of the oscillations of the internal mass. This talk drawn from the results in [5] and other recent research will present a toy model of a bristle bot made from a toothbrush, and discuss a theoretical model for its dynamics and show that its dynamics can be classified into four modes of motion: purely stick (no locomotion), slip, stick-slip and hopping. In the stick mode, the dynamics of the system are those of a nonlinear Mathieu oscillator and large amplitude resonance oscillations lead to the slip mode of motion. The mode of motion depends on the amplitude and frequency of the periodic forcing. A phase diagram can be computed that captures this behavior, that is reminiscent of the tongues of instability seen in a Mathieu oscillator. The broader result that emerges paper is that it may be possible to actively tune the frictional interaction between a robot and the substrate on which the robot is moving purely through internal shape variables to produce agile and controllable locomotion. Some experimental results on the motion of such robots in pipes and restrictive environments will be also be discussed as possible applications.

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On the isochronicity problem

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Our main result is the complete set of explicit conditions necessary and sufficient for isochronicity of a Hamiltonian system with one degree of freedom. The conditions are presented in terms of Taylor coefficients of the Hamiltonian function.

On completely noninvariant Killing tensors in Euclidean geometry

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According to Erlangen Program the main goal of any branch of geometry is to study the manifold configurations with respect to those features that are not altered by the transformations of the group. Because isometry group and the overwhelming majority of the corresponding invariants in Euclidean geometry are well studied, we will discuss completely non-invariant objects.

Invariant classification of the Killing tensors of valency two allows us to construct quadratic integrals of motion and to relate the corresponding integrable Stäckel systems with conic section theory, Abel's quadratures, stationary flows of KdV and other nonlinear equations, algebro-geometric constructions of curvilinear coordinates, Turiel's deformations of the Poisson brackets, Gaudin magnets, black holes classification, Nijenhuis – Haantjes geometry, Kohno – Drinfeld Lie algebras, Stasheff politopes, etc.

Completely nonivariant with respect to the action of the isometry group Killing tensors did not arise in any of these theories. We want to fill this gap by starting, as in the invariant theory, with the Darboux type calculations in three-dimensional Euclidean space [1]. The first results in this direction will be discussed.

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Motion of a circular disk in the presence of a point source in an ideal fluid

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Consider the plane-parallel motion of an unbalanced circular foil in the presence of a point source in an unbounded volume of an ideal incompressible fluid. We assume that the fluid performs potential noncircular motion and is at rest at infinity.

To describe the motion of the system, we introduce three coordinate systems: a fixed (inertial) system OXY, a moving system $Cx'y'$, rigidly attached to the foil, and a coordinate system Oxy rotating synchronously with the foil (see Fig. 1). We will assume that the origin of the moving coordinate system C is at the geometric center of the foil and that the center of mass of the foil lies on the positive part of the axis Cx'.

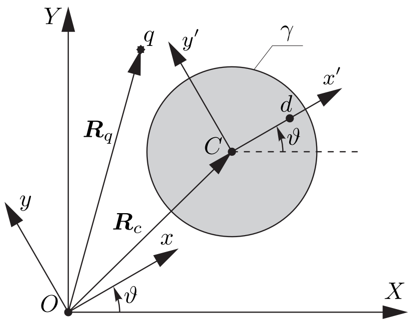

Fig. 1. A schematic diagram of an unbalanced circular foil and a point source.

In the case of fixed source of constant intensity the equation of motion can be represented in Hamiltonian form

$$\begin{gathered}\dot{X}_c = \frac{\partial H}{\partial \Pi_x}, \quad \dot{Y}_c = \frac{\partial H}{\partial \Pi_y}, \quad \dot{\vartheta} = \frac{\partial H}{\partial \Pi_\vartheta}, \\ \dot{\Pi}_x = -\frac{\partial H}{\partial X_c}, \quad \dot{\Pi}_y = -\frac{\partial H}{\partial Y_c}, \quad \dot{\Pi}_\vartheta = -\frac{\partial H}{\partial \vartheta},\end{gathered} \tag{1}$$

where the Hamiltonian H is given by the following expression:

$$H = \frac{1}{2}\big(\boldsymbol{\mathcal{P}}, \mathbf{Q}^{-1}\boldsymbol{\mathcal{P}}\big) - \frac{\rho q^2}{4\pi}\Big(\ln\big(X_c^2 + Y_c^2\big) - \ln\big(X_c^2 + Y_c^2 - R^2\big)\Big), \tag{2}$$

$$\boldsymbol{\mathcal{P}} = \begin{pmatrix} \Pi_x + A_x \\ \Pi_y + A_y \\ \Pi_\vartheta \end{pmatrix}, \quad \mathbf{Q} = \begin{pmatrix} m_c + \rho\pi R^2 & 0 & -m_c d\sin\vartheta \\ 0 & m_c + \rho\pi R^2 & m_c d\cos\vartheta \\ -m_c d\sin\vartheta & m_c d\cos\vartheta & I_c + m_c d^2 \end{pmatrix}, \tag{3}$$

$$A_x = \frac{\rho q R^2 X_c}{X_c^2 + Y_c^2}, \quad A_y = \frac{\rho q R^2 Y_c}{X_c^2 + Y_c^2}. \tag{4}$$

Equations (1) admit two first integrals: an energy integral coinciding with the Hamiltonian (2) and the integral of the angular momentum

$$K = \Pi_\vartheta + \Pi_y X_c - \Pi_x Y_c = \mathrm{const}, \tag{5}$$

which is a consequence of the existence of a symmetry field

$$\boldsymbol{u} = -Y_c\frac{\partial}{\partial X_c} + X_c\frac{\partial}{\partial Y_c} + \frac{\partial}{\partial\vartheta} - \Pi_y\frac{\partial}{\partial\Pi_x} + \Pi_x\frac{\partial}{\partial\Pi_y}. \tag{6}$$

Using two consecutive changes of variables

$$\begin{gathered} p_x = (\Pi_x + A_x)\cos\vartheta + (\Pi_y + A_y)\sin\vartheta, \\ p_y = -(\Pi_x + A_x)\sin\vartheta + (\Pi_y + A_y)\cos\vartheta, \\ p_\vartheta = \Pi_\vartheta, \quad x = X_c\cos\vartheta + Y_c\sin\vartheta, \quad y = -X_c\sin\vartheta + Y_c\cos\vartheta \end{gathered} \tag{7}$$

and

$$x = r\cos\varphi, \quad y = r\sin\varphi, \quad p_x = p\cos\alpha, \quad p_y = p\sin\alpha, \tag{8}$$

we can obtain the following reduced system

$$\begin{gathered} \dot r = m^{-1}p\cos(\alpha-\varphi) - m^{-1}m_c d\sin\varphi\Omega, \\ \dot\varphi = \frac{p\sin(\alpha-\varphi)}{mr} - \frac{m_c d\cos\varphi}{mr}\Omega - \Omega, \\ \dot p = -\frac{\rho q^2R^2}{2\pi}\frac{\cos(\alpha-\varphi)}{r(r^2-R^2)}, \quad \dot\alpha = \frac{\rho q^2 R^2}{2\pi}\frac{\sin(\alpha-\varphi)}{r(r^2-R^2)p} - \Omega, \end{gathered} \tag{9}$$

$$\Omega = \frac{k - rp\sin(\alpha-\varphi) - m^{-1}m_c dp\sin\alpha}{I_c + m_c d^2 - m^{-1}m_c^2 d^2}. \tag{10}$$

Nonintegrability of the system consider is shown using the scattering map [1], see. Fig. 2.

More detailed description of this investigation can be found in Ref. [2].

The work was carried out within the framework of the state assignment of the Ministry of Education and Science of Russia (FEWS-2020-0009).

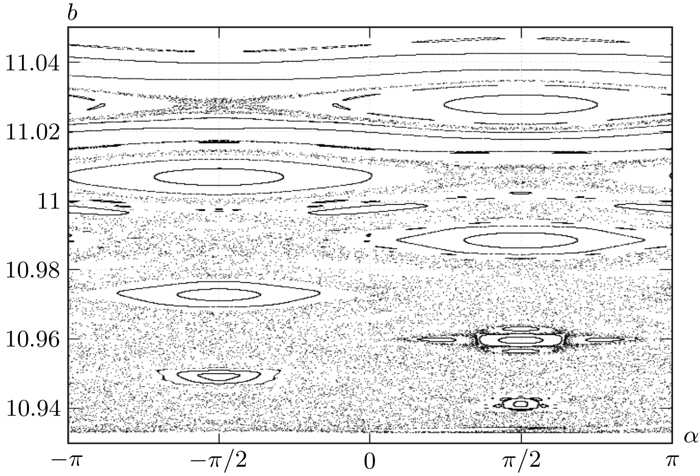

Fig. 2. The parameter values $m_c = 1$, $\rho = 1$, $d = 0.01$, $I_c = 1$, $R = 1$, $k = 1$, $q = 1$, $h = 0.001$, $r_{max} = 100$. As the initial momentum we have chosen the largest value. Here $b = r\sin(\alpha - \varphi) + m^{-1}m_c d \sin\alpha$.

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