Dynamic interaction of point vortices and a two-dimensional cylinder

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In this paper we consider the system of an arbitrary two-dimensional cylinder interacting with point vortices in a perfect fluid. We present the equations of motion and discuss their integrability. Simulations show that the system of an elliptic cylinder (with nonzero eccentricity) and a single point vortex already exhibits chaotic features and the equations of motion are nonintegrable. We suggest a Hamiltonian form of the equations. The problem we study here, namely, the equations of motion, the Hamiltonian structure for the interacting system of a cylinder of arbitrary cross-section shape, with zero circulation around it, and \( N \) vortices, has been addressed by Shashikanth [Regular Chaotic Dyn. 10, 1 (2005)]. We slightly generalize the work by Shashikanth by allowing for nonzero circulation around the cylinder and offer a different approach than that by Shashikanth by using classical complex variable theory.

I. INTRODUCTION

In this paper we consider the system of an arbitrary two-dimensional cylinder interacting with point vortices in a perfect fluid. We present the equations of motion and discuss their integrability. Simulations show that the system of an elliptic cylinder (with nonzero eccentricity) and a single point vortex already exhibits chaotic features and the equations of motion are nonintegrable. We suggest a Hamiltonian form of the equations. The problem we study here, namely, the equations of motion, the Hamiltonian structure for the interacting system of a cylinder of arbitrary cross-section shape, with zero circulation around it, and \( N \) vortices, has been addressed by Shashikanth [Regular Chaotic Dyn. 10, 1 (2005)]. We slightly generalize the work by Shashikanth by allowing for nonzero circulation around the cylinder and offer a different approach than that by Shashikanth by using classical complex variable theory.

II. FLOW AROUND A MOVING CONTOUR

In this paper only two-dimensional hydrodynamic problems are discussed. Consider a two-dimensional neutrally buoyant rigid body whose boundary, \( C \), is not necessarily a circle. We assume that the region enclosed by \( C \) is simply connected and \( C \) itself is a Jordan curve. This guarantees that the exterior of \( C \) can be conformally mapped onto the exterior of a unit disk;
Obviously, by virtue of the Cauchy-Riemann equations one can express the Dirac delta function. The impermeable boundary conditions on the contour \( C \) of \( \Gamma_a \) and vice versa should be mentioned. He proved that the equations of motion for vortices in the absence of any walls and inside or outside a circular cylinder are Hamiltonian with the same bracket.

Let \( O_x y \) be fixed in space laboratory frame of reference. We fix another orthogonal frame \( O_x \xi \eta \) to the body (Fig. 1). The position of a point relative to the laboratory frame and the body-fixed frame will be represented as \( z = x + iy \) and \( \xi = \xi + i \eta \), respectively. The location of the contour \( C \) relative to the laboratory frame is well defined by the coordinates \( z_0 = x_0 + iy_0 \) of the origin \( O_x \), and the angle of rotation \( \theta \). The complex numbers \( z_\alpha, \alpha = 1, \ldots, n \), describe the positions of the point vortices.

It is well known that the velocity of the fluid is given by a multivalued potential \( \varphi(z) \) or a stream function \( \varphi(z) \). The velocity at a point \( z \) reads

\[
\mathbf{u}(z) = u_x' + iu_y' = \frac{\partial \varphi}{\partial x} + i \frac{\partial \varphi}{\partial y} = \frac{\partial \varphi}{\partial y} - i \frac{\partial \varphi}{\partial x}.
\]

Obviously, by virtue of the Cauchy-Riemann equations one can express \( \varphi \) in terms of quadratures of \( \psi \) (and vice versa). Using the complex potential \( w(z) = \varphi(z) + i \psi(z) \), we can find the velocity as \( \mathbf{u} = \partial w / \partial z \) (the bar denotes the complex conjugate).

Due to the point vortices the stream function satisfies the equation

\[
\Delta \varphi(z) = \sum_{\alpha}^{n} \Gamma_{\alpha} \delta(z - z_\alpha),
\]

where \( z_\alpha \) and \( \Gamma_{\alpha} \) are the coordinates and intensities of the vortices; \( \delta(z - z_\alpha) = \delta(x - x_\alpha) \delta(y - y_\alpha) \) is the Dirac delta function. The impermeable boundary conditions on \( C \) and the zero velocity at infinity (see Refs. 6 or 8) imply that

\[
\varphi|_C = u_x y - u_y x + \frac{\omega}{2} (x - x_0)^2 + (y - y_0)^2, \quad \varphi(\infty) = 0.
\]

Here \( u_x = \dot{x}_0, \ u_y = \dot{y}_0 \), and \( \omega = \dot{\theta} \) are the linear and angular velocities of the body. The circulation around the body can be written as

\[
\oint_C (\mathbf{u}, dl) = \oint_C d\varphi = \oint_C \frac{\partial \psi}{\partial n} dl = \Gamma_c = \text{const},
\]

where \( dl \) is an element of the contour’s arc length. It is well known (see, for example, Refs. 6 or 8) that the stream function can be represented in the form

![Fig. 1. Definition of the coordinates of body and vortices.](image-url)
\[ \psi(z) = v_z \psi_z(z) + v_y \psi_y(z) + \omega \psi_u(z) + \Gamma^* \psi_0(z) + \sum_a \Gamma_a \psi_a^*(z). \]  

(1)

Here all the terms (except for \( \psi_a^* \), the stream function due to the \( a \)th vortex) are harmonic functions in the domain exterior to \( C \), decay at infinity, and satisfy the following conditions:

\[ \psi_z|_C = -y, \quad \psi_y|_C = x, \quad \psi_u|_C = \frac{(x-x_0)^2 + (y-y_0)^2}{2}, \quad \psi_0|_C = \text{const}. \]

For the circulation we can write

\[ \oint_C \mathrm{d} \varphi_x = \oint_C \mathrm{d} \varphi_y = \oint_C \mathrm{d} \varphi_u = 0, \quad \oint_C \mathrm{d} \varphi_0 = \Gamma^* = \Gamma_C + \sum \Gamma_a. \]

**Remark:** The circulation \( \Gamma^* \) is introduced for convenience to represent the total circulation about the body minus the circulation induced by the vortices. Hereafter, the constant \( \Gamma^* \) will be referred to as pure circulation. The pure circulatory flow around the cylinder (the flow which remains when the cylinder’s velocity and the point vortices vanish) is governed by the stream function \( \psi_0(z) \).

All the functions \( \psi_a^* \) can be expressed in terms of single Green’s function which satisfies the equation

\[ \Delta G(z,z_0) = \delta(z-z_0), \quad G(z,z_0)|_{z \in C} = \text{const}, \]

therefore \( \psi_a^*(z) = G(z,z_0) \).

We will show now that the other functions from Eq. (1) can also be found in terms of quadratures of \( G(z,z_0) \). First, let us note that the functions possess the following properties.

1. The function \( \psi_a^*(z) \) allows the following representation:

\[ \psi_a^*(z) = -\frac{1}{2\pi} \log|z-z_a| + \psi_a^0(z), \]

where \( \psi_a^0(z) \) is that part of \( \psi_a^* \) which is regular at the point \( z=z_a \), in other words \( \psi_a^0(z) \) is harmonic outside \( C \) and \( \psi_a^0(\infty) = 0 \). Therefore, as \( z_a \to \infty \) we have \( \psi_a^0(z) \to \psi_0(z) \).

2. The solution to the boundary value problem with Dirichlet conditions, that is, \( \Delta \Phi(z) = 0, \quad \Phi|_C = f(z) \), can be represented using Green’s function (2) as follows:

\[ \Phi(z) = \oint_C f(\xi) \frac{\partial G}{\partial n}(\xi,z) \mathrm{d}T. \]

Therefore, we get

\[ \psi_z(z_a) + i \psi_y(z_a) = i \oint_C \frac{\partial \psi_a^0}{\partial n} \mathrm{d}T, \quad \psi_u = \oint_C \frac{|z-z_0|^2}{2} \frac{\partial \psi_a^0}{\partial n} \mathrm{d}T. \]

Thus, to determine the flow around the body we need to know only the function \( G(z,z_0) \). Given a conformal mapping of the region exterior to \( C \) onto the exterior of the unit disk \( \zeta = \mathcal{F}(z) \), one readily finds Green’s function to be

\[ G(z,z_0) = \text{Re} \left( \frac{1}{2\pi i} \log(\mathcal{F}(z) - \mathcal{F}(z_0)) - \frac{1}{2\pi i} \log \left( \mathcal{F}(z) - \frac{1}{\mathcal{F}(z_0)} \right) \right). \]

Expanding the inverse mapping \( z = \mathcal{F}^{-1}(\zeta) \) into a Laurent series, one gets
\[ F^{-1}(z') = kz' + z_0 + \frac{k_1}{z'} + \frac{k_2}{z'^2} + \ldots, \]

where \( k \) is some positive real number. The point \( z_0 = x_0 + iy_0 \), often referred to as the contour’s conformal center, is fixed in the body and can be found from the equation

\[ z_0 = \frac{1}{2\pi i} \oint_{C} \frac{dz'}{z'}. \]

The contour’s conformal center coincides with the point of application of the Zhukovsky lifting force.\(^6\)

### III. Equations of Motion for the Contour

To obtain the equations, we start with calculating the force and moment acting on the body. According to Ref. 4, the force acting on the body in the presence of point vortices looks like

\[ f = f_x + if_y = \frac{d}{dt} \oint_C iz \frac{\partial \psi}{\partial n} - i \sum_a \Gamma_a v_a, \]

where \( v_a = \dot{z}_a \) is the velocity of the point vortex with coordinate \( z_a \) and \( \psi \) is the stream function given by Eq. (1).

We see that the force depends linearly on the elementary stream functions from Eq. (1). Let us see how each elementary function contributes to the force.

1. As it was shown by Kirchhoff, the elementary functions that determine the fluid’s kinetic energy for \( \Gamma^x = 0, \Gamma^a = 0 \) produce the well-known added-mass effect,

\[ \frac{d}{dt} \int_C iz \frac{\partial}{\partial n}(v_x \psi_x + v_y \psi_y + \omega \psi_w) dl = - \frac{d}{dt} \left( \frac{\partial T^x}{\partial v_x} + i \frac{\partial T^y}{\partial v_y} \right). \]

Here

\[ T^x_0 = \frac{1}{2} \int_{R^2} (v_x^2 + v_y^2) |_R^x = 0, \Gamma^a = 0 dx dy = - \frac{1}{2} \oint_C \varphi d \psi |_{R^x = 0, \Gamma^a = 0} \]

\[ = \frac{1}{2} (a_{11} v_x^2 + a_{22} v_y^2 + 2a_{12} v_x v_y + b \omega^2 + 2 c_1 v_x \omega + 2 c_2 v_y \omega). \]

The coefficients \( a_{ij}, b, c_i \) are the body’s added masses and moments, which generally depend on the angle of rotation \( \theta \).

2. The body is also subject to the lifting force caused by the purely circulation flow \( (\Gamma^x \neq 0, \Gamma^a = 0, v_x = v_y = \omega = 0) \). By Sedov’s theorem,\(^6\) in this case the body is acted on by an additional Zhukovsky lifting force \( i l \Gamma^x (dz_0/dt) \), which is applied to the body’s conformal center.

3. The contribution due to the point vortices is\(^4\)

\[ f_a = \Gamma_a \frac{d}{dt} \oint_C iz \frac{\partial \psi^a}{\partial n} dl + i \Gamma_a v_a. \]

The contour integral can be easily calculated using property 2. We have

\[ f_a = \frac{d}{dt} \Gamma_a (\dot{\psi}_x(z_a) + i \dot{\psi}_y(z_a) + iz_a). \]

Thus, the equation of motion for the body in the fixed frame of reference looks like
The fluid's velocity becomes
\[ \vec{u}(\xi) = u(\psi_0(\xi) - \eta) + v(\psi_0(\xi) + \xi) + \omega(\psi_0(\xi) + \frac{1}{2}i|\xi|^2) + \Gamma_0^I \psi_0(\xi) + \sum \Gamma_\alpha \psi_\alpha(\xi), \]
where \( u \) and \( v \) are the components of the absolute velocity of the point \( O_c \) in the body-fixed frame. Therefore,
\[ \psi_0 = \psi_x \cos \theta + \psi_y \sin \theta, \quad \psi_c = -\psi_y \sin \theta + \psi_x \cos \theta. \]
The fluid's velocity becomes \( \vec{u}' = (\partial \vec{u}/\partial \eta, -\partial \vec{u}/\partial \xi) \). Let
\[ \vec{\psi}_\alpha(\xi) = \vec{\psi}(\xi) + \frac{\Gamma_\alpha}{2\pi} \log|\xi - \zeta_\alpha|, \quad \alpha = 1, \ldots, n. \]
Each \( \vec{\psi}_\alpha(\xi) \) is analytic at the point coinciding with the \( \alpha \)th vortex. Therefore, the velocity of the vortices in the body-fixed frame can be written as \( \vec{\xi}_\alpha = \partial \vec{\psi}_\alpha(\xi)/\partial \eta - i(\partial \vec{\psi}_\alpha(\xi)/\partial \xi)|_{\xi=\zeta_\alpha}. \)
Shifting the origin of the body-fixed frame to the conformal center and using vector notation, we arrive at the following theorem.

**Theorem:** The equations of motion of the body and vortices in the body-fixed frame can be written in Kirchhoff's form as follows:

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial V} - \sum \Gamma_\alpha \frac{\partial \vec{\psi}_\alpha(\xi_\alpha)}{\partial V} \right) + \Omega \times \left( \frac{\partial T}{\partial V} - \sum \Gamma_\alpha \frac{\partial \vec{\psi}_\alpha(\xi_\alpha)}{\partial V} \right) = \Gamma^* k \times V, \]

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \omega} - \sum \Gamma_\alpha \frac{\partial \vec{\psi}_\alpha(\xi_\alpha)}{\partial \omega} \right) k + V \times \left( \frac{\partial T}{\partial V} - \sum \Gamma_\alpha \frac{\partial \vec{\psi}_\alpha(\xi_\alpha)}{\partial V} \right) = 0, \]

\[ \frac{\partial \vec{\psi}_\alpha(\xi)}{\partial \vec{\zeta}} \bigg|_{\xi=\zeta_\alpha} = i \frac{\partial \vec{\psi}_\alpha(\xi)}{\partial \xi}, \quad \alpha = 1, \ldots, n, \]

\[ \dot{x}_0 = u \cos \theta - v \sin \theta, \quad \dot{y}_0 = u \sin \theta + v \cos \theta, \quad \dot{\theta} = \omega, \]
Here \( x_0 \) and \( y_0 \) are the coordinates of the conformal center in the fixed frame of reference, \( \theta \) is the angle of rotation that describes the position of the body-fixed frame relative to the laboratory frame, and \( T \) is the kinetic energy of the body + fluid system in the absence of the vortices and circulation around the body \( (\Gamma^+ = \Gamma^- = 0) \). Hereafter, we adopt the following convention: by the derivation of a real-valued function \( \tilde{\psi}_a(\zeta) \) with respect to a complex variable \( \zeta = \xi + i \eta \) we mean \( \partial \tilde{\psi}_a(\zeta) / \partial \zeta = \tilde{\psi}_a(\zeta) / \partial \xi + i \tilde{\psi}_a(\zeta) / \partial \eta \).

**Remark:** In the general case of asymmetric body, the kinetic energy is a quadratic form whose matrix is not diagonal. If \( O_e \) is not at the conformal center, then additional terms linear in \( V, \omega \), and proportional to \( \Gamma^+ \) appear in the right-hand side of the first three equations [Eqs. (9)].

**Remark:** Particular cases of Eqs. (9) have been obtained in Refs. 1–4. The case of a circular cylinder has been considered in greater detail in Refs. 1, 2, and 4. In Ref. 5, it has been shown that the equations of motion for the system of a circular cylinder and a single vortex are integrable.

### IV. POISSON STRUCTURE AND INTEGRALS OF MOTION

Equations (9) can be represented in a “nearly” Lagrangian form. As a counterpart of the Lagrangian function, we take

\[
L = T - \sum_{a=1}^{n} \Gamma_a \left( \psi_a(\xi_0) - \eta_0 \right) u + \left( \psi_a(\xi_0) + \xi_0 \right) v + \left( \psi_a(\xi_0) - \frac{1}{2} |\xi_0|^2 \right) \omega - \Gamma^+ \sum_{a=1}^{n} \Gamma_a \psi_a(\xi_0)
- \frac{1}{4 \pi} \sum_{a=1}^{n} \Gamma_a \log \left| \frac{\mathcal{F}(\xi_0)}{1 - |\mathcal{F}(\xi_0)|^2} \right| - \frac{1}{4 \pi} \sum_{a \neq \beta} \Gamma_a \Gamma_{\beta} \log \left| \frac{\mathcal{F}(\xi_0) - \mathcal{F}(\xi_{\beta})}{1 - \mathcal{F}(\xi_0) \mathcal{F}(\xi_{\beta})} \right|
\]

(10)

where \( \mathcal{F}(\zeta) \) maps conformally the domain exterior to \( C \) onto the exterior of the unit disk. Then the equations of motion take the form

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{V}} \right) + \Omega \times \frac{\partial L}{\partial \dot{V}} = \Gamma^+ \dot{k} \times V, \quad \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{V}} \right) \dot{k} + V \times \frac{\partial L}{\partial \dot{V}} = 0,
\]

(11)

Performing a Legendre transformation of the system (11), we get

\[
P = \frac{\partial L}{\partial V} = \frac{\partial T}{\partial V} \sum_a \Gamma_a \frac{\partial \tilde{\psi}_a(\xi_0)}{\partial V}, \quad M = \frac{\partial L}{\partial \omega} = \frac{\partial T}{\partial \omega} \sum_a \Gamma_a \frac{\partial \tilde{\psi}_a(\xi_0)}{\partial \omega},
\]

\[
H = (P, V) + M \omega - L|_{\dot{V} = P, \dot{\omega} = M}, \quad P = (P_u, P_v, 0).
\]

(12)

Next, we introduce the following notation: \( W = (u, v, \omega) \) and \( \pi = (P_u, P_v, M) \). Assume that in the absence of circulation the kinetic energy is a homogeneous function of \( u, v, \) and \( \omega \), that is,

\[
T = \frac{1}{2} (A W, W).
\]

The components of the matrix \( A \) involve the added masses and moments. The Hamiltonian (12) takes the form
where the Hamiltonian function is given by Eq. (13) and the Poisson brackets are as follows:

\[
\{ P_x, P_y \} = -1, \quad \{ M, P_x \} = -P_v, \quad \{ M, P_y \} = P_u,
\]

\[
\{ \xi, \eta \} = \Gamma^{-1}, \quad \{ x_0, P_u \} = \cos \theta, \quad \{ x_0, P_v \} = -\sin \theta,
\]

\[
\{ y_0, P_u \} = \sin \theta, \quad \{ y_0, P_v \} = \cos \theta, \quad \{ \theta, M \} = 1.
\]

Remark: Historically, the following special cases of the system (14) and (15) have been considered: 1. \( \Gamma_i = 0 \) for all \( i \) but \( \Gamma^e \neq 0 \). This case was investigated already by Chaplygin. The case of a circular cylinder with \( \Sigma \Gamma_i = 0, \Gamma^e = 0 \), and arbitrary \( i \) has been treated in Ref. 2. The equations of motion of a circular cylinder for arbitrary \( \Sigma \Gamma_i \) and \( \Gamma^e \) were obtained in Ref. 4.

V. INTERACTION OF AN ELLIPTIC CYLINDER WITH A SINGLE POINT VORTEX

To find the stream function \( \tilde{\psi}(\zeta) \), we map conformally \( (\zeta' = F(\zeta)) \) the exterior of the ellipse onto the exterior of the circle of unit radius on the complex plane \( \zeta' = \xi' + i \eta' \).
\[ \zeta = \mathcal{F}^{-1}(\zeta') = \frac{a + b}{2} \zeta' + \frac{a - b}{2} \frac{1}{\zeta'}, \]

where \( a \) and \( b \) are the semiaxes of the ellipse. Using the classical circle theorem, we find the stream function to be

\[ \tilde{\psi}(\xi) = u(\psi_u(\xi) - \eta) + v(\psi_v + \xi) + \omega \left( \psi_u(\xi) - \frac{1}{2} |\xi|^2 \right) - \frac{\Gamma^*}{2\pi} \log|\mathcal{F}(\xi)| + \frac{\Gamma_1}{2\pi} \log \left| \mathcal{F}(\xi) - \frac{1}{\mathcal{F}(\xi_1)} \right| \]

\[ - \frac{\Gamma_1}{2\pi} \log[\mathcal{F}(\xi) - \mathcal{F}(\xi_1)]. \]  

(19)

Here \( \xi_1 = \xi_1 + i \eta_1 \) denotes the position of the vortex and

\[ \psi_u = -b \frac{\eta'}{\xi^2 + \eta'^2}, \quad \psi_v = -a \frac{\xi'}{\xi^2 + \eta'^2}, \quad \psi_w = \frac{b^2 - \omega^2}{4} \frac{\xi'^2 - \eta'^2}{(\xi^2 + \eta^2)^2}. \]

Substituting these expressions into Eqs. (13) and (14) yields a closed system of equations in the components of the generalized momentum \( P = (P_u, P_v, M) \) and the coordinates of the vortex \((\xi_1, \eta_1)\). The system admits two integrals of motion: the energy and \( F \) [Eq. (18)]. On a level surface of the integral (18), one gets a Hamiltonian system with two degrees of freedom. A three-dimensional (3D) view of the Poincaré section in the space of \( P_u, P_v, \) and \( M \) and its two-dimensional (2D) projections are shown in Figs. 2 and 3.

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**FIG. 2.** Poincaré section of the system in the 3D-space of \( P_u, P_v, M \).
One can see that for $a/b = \frac{1}{2}$ there are regions of chaos which indicates the nonintegrability of the system (in contrast to the circular case $a=b$, which is integrable). It seems to be interesting to explore how the integrability and chaotic behavior of the system vary with the value of $a/b$.

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