INTEGRABLE AND NON-INTEGRABLE HAMILTONIAN SYSTEMS

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Abstract

Various effects of a qualitative kind that prevent the integrability of Hamiltonian systems are considered. The paper is a continuation of a previous survey given by the author (Usp. Mat. Nauk, 38, No. 1, 1983) but also provides an independent treatment whose main aim is to survey the results obtained on this topic since 1983.
INTRODUCTION

In the last 10–15 years mathematicians have found a new interest in topics connected with the integration of the equations (usually Hamiltonian) of classical dynamics. New completely integrable systems (including multi-dimensional analogues of classical problems) have been found, and various algebraico-geometric constructions that make clear the reasons for the existence of "latent" conservation laws have been proposed. In addition, it has been found useful to examine the features of the behaviour of the phase trajectories of non-integrable Hamiltonian systems and to give a strict proof of their non-integrability. The present paper deals with various effects of a qualitative kind that prevent the integrability of Hamiltonian systems. While the paper can be regarded as a continuation of the author's survey [1], it offers an independent treatment whose main aim is to acquaint the reader with the results obtained on this topic since 1983.

1. We first recall the definition of Hamiltonian dynamic system. Let $M^{2n}$ be an even-dimensional manifold (phase space), $\omega$ a closed non-degenerate 2-form in $M$ (simplectic structure), $H$ a real function in $M$ (Hamiltonian function). Since $\omega$ is non-degenerate, the function $H$ can be associated with a unique vector field $v_H$, given by the equation

$$\omega(v_H, \cdot ) = dH.$$  

This field generates a Hamiltonian system in $M$:

$$\dot{x}(t) = v_H(x(t)), \quad x : \mathbb{R} \rightarrow M.$$  \hspace{1cm} (0.1)

In suitable so-called canonical local coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$ the form $\omega$ reduces to the form $\Sigma dx_s \wedge dy_s$ (Darboux theorem). In canonical coordinates $x, y$ the Hamiltonian equations (0.1) have the more usual appearance

$$\dot{x}_s = - \frac{\partial H}{\partial y_s}, \quad \dot{y}_s = \frac{\partial H}{\partial x_s}; \quad 1 \leq s \leq n.$$  \hspace{1cm} (0.2)

It is often a question of considering non-autonomous Hamiltonian systems in which the Hamiltonian $H$ depends explicitly on time.

If the differential equations, written in local coordinates in $M$, do not have the form (0.2), this does not necessarily imply that they are not Hamiltonian. Let us give some examples of dynamical systems whose Hamiltonian properties are not obvious a priori.
(a) Our first example is the linear system with constant coefficients
\[ \dot{x} = Ax, \quad x \in \mathbb{R}^n \]  
which has the first integral \( f = (Bx, x)/2 \), where \( B \) is a non-degenerate symmetric operator.

**Theorem 1.** If \( \det A \neq 0 \), then

1. \( (\mathbb{R}^n, \omega), \omega(x', x'') = (BA^{-1}x', x'') \) is a simplectic manifold,
2. the vector field \( Ax \) is Hamiltonian with Hamilton function \( f \).

**Proof.** Since \( f \) is an integral of Eqs. (0.3), then \( \dot{f} = (Bx, Ax) = (x, BAx) = 0 \). Consequently, \( BA \) is a skew-symmetric operator. Hence, in turn, the operator \( BA^{-1} \) is skew-symmetric. Since \( A \) and \( B \) are non-degenerate, the outer 2-form \( \omega \) is likewise non-degenerate. Like every outer form with constant coefficients, this form is closed. It remains to observe that
\[ \omega(Ax, \cdot) = (BA^{-1}(Ax), \cdot) = (Bx, \cdot) = df. \]

(b) We take Poisson's equation of rigid body dynamics
\[ \dot{e} = e \times \dot{\omega} \]
where \( e \) and \( \dot{\omega} \) are vectors of three-dimensional oriented Euclidean space, \( \dot{\omega} \) being a known function of time. Equation (0.4) has the integral \( (e, e) = c \geq 0 \). Since \( e \) is a unit vector in rigid body dynamics, we put \( c = 1 \). We furnish the sphere \( S^2 = \{e: (e, e) = 1\} \) with a simplectic structure, by putting \( \omega(x', x'') = (e, x' \times x'') \), where \( x' \) and \( x'' \) are tangent vectors to \( S^2 \) at the point \( e \). The form \( \omega \), which is an oriented area of \( S^2 \), is closed and non-degenerate, but not exact:
\[ \int_{S^2} \omega = 4\pi. \]
The vector field \( v = e \times \dot{\omega}(t) \) is a non-stationary tangent field in \( S^2 \).

**Theorem 2.** Equation (0.4) is a Hamilton equation in the simplectic manifold \( (S^2, \omega) \) with Hamiltonian \( H = -(e, \dot{\omega}(t)) \).

For, \( \omega(v, \cdot) = (e, (e \times \dot{\omega}) \times (\cdot)) = (\cdot, e \times (e \times \dot{\omega})) = (\cdot, e(\dot{\omega}, e) - \dot{\omega}) = - (\cdot, \dot{\omega}) = dH. \)
An interesting point is that, if \( e = \dot{\xi}(t) \) is the solution of Eqs. (0.4), then the function \( f = (\dot{\xi}(t), e) \) is its first integral. If \( \dot{\omega} \) is a \( p \)-periodic function of time, then a mapping over the period of the linear system (0.4) preserves the oriented area \( S^2 \) and hence has at least two distinct fixed points. In this case there is an integral which is \( p \)-periodic in \( t \).

(c) Following Birkhoff, we also consider the "generalized Pfaff problem" on the stationary curves of the functional

\[
P(x(\cdot)) = \int_{t_1}^{t_2} (\sum u_i \dot{x}_i + B) \, dt.
\]

Here, \( u_i \) and \( B \) are smooth functions of variables \( x_i, \ldots, x_n \) and \( t \) (see [2, 3]). It is easily shown that the variational equation \( \delta P = 0 \) \( (\delta x_i(t_1) = \delta x_i(t_2) = 0) \) defines the variables \( x_i \) as functions of \( t \) which satisfy the system of equations

\[
\frac{\partial u}{\partial t} + (\text{rot } u) \dot{x} = \frac{\partial B}{\partial x}, \quad \text{rot } u = \left\| \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right\|.
\]

The matrix \( \text{rot } u \) is assumed to be non-degenerate for all the considered values of variables \( x_i \) and \( t \). Thus \( n \) is even and Eqs. (0.5) uniquely define non-stationary vector field in variables \( x_i, \ldots, x_n \).

**Theorem 3 ([4]).** In a suitable simplectic structure, Eqs. (0.5) are Hamiltonian with Hamilton function \( B \).

If, for example, the \( u_j \) are not explicitly dependent on \( t \), then Eqs. (0.5) are Hamiltonian in the simplectic manifold \( (\mathbb{R}^n = \{x_i\}, \omega) \), \( \omega = d(\sum u_j dx_j) \). A description of the simplectic structure in the non-autonomous case is to be found in [4]. The statement of the problem as to whether Eqs. (0.5) are Hamiltonian is due to Birkhoff. He examined the reduction of (0.5) to the canonical form in a neighbourhood of the equilibrium position with the aid of formal replacement of the independent variables [2].

(d) The problem of the rotation of a rigid body in a homogeneous magnetic field in the context of magnetization effects of the Barnett-London type, see e.g., [5]. The equations of rotation reduce to the system

\[
I \ddot{\omega} + \dot{\omega} \times I \dot{\omega} = (\Lambda \dot{\omega}) \times e, \quad \dot{e} + \dot{\omega} \times e = 0.
\]
Here, \( I \) is the inertia operator, while \( \Lambda \) is a symmetric operator characterizing the body magnetization properties. If \( \Lambda = \lambda E \), then Eqs. (0.6) reduces to Kirchhoff's equations of the problem on the motion of a rigid body in an ideal fluid, the Hamiltonian nature of which is well known, see e.g., [6, 7]. For, on introducing the function

\[
L = \frac{1}{2} (I \dot{\omega}, \dot{\omega}) + \lambda (\dot{\omega}, e),
\]

we can write Eqs. (0.6) in the form

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{\omega}} + \dot{\omega} \times \frac{\partial L}{\partial \dot{\omega}} = \frac{\partial L}{\partial e} \times e, \quad \dot{e} + \dot{\omega} \times e = 0.
\]

These equations can be treated as Poincaré's equations in the group \( SO(3) \) with Lagrangian \( L \) (regarding Poincaré's equations see e.g., [8]). Let us write Poincaré's equations as Hamilton equations. For this, we put

\[
m = \frac{\partial L}{\partial \dot{\omega}} = I \dot{\omega} + \lambda e
\]

and introduce the Hamilton function

\[
H(m, e) = (m, \dot{\omega}) - L(\dot{\omega}, e)|_{m, e}
\]

Invariables \( m, e \), Eqs. (0.6) take the form of Kirchhoff's equations

\[
\dot{m} = m \times \frac{\partial H}{\partial m} + e \times \frac{\partial H}{\partial e}, \quad \dot{e} = e \times \frac{\partial H}{\partial m}
\]

with the Hamilton function

\[
H = \frac{1}{2} (I^{-1}m, m) - \lambda (I^{-1}m, e) + \frac{\lambda^2}{2} (I^{-1}e, e).
\]

It was shown in [5] that in this case Eqs. (0.6) do not have an analytic integral, independent of the "classical" integrals \( F_1 = (I \dot{\omega}, \dot{\omega})/2 \), \( F_2 = (I \dot{\omega}, e) \), \( F_3 = (e, e) \), if the eigenvalues of the inertia operator are distinct. If a pair of eigenvalues coincides, Eqs. (0.6) have an auxiliary integral which is linear in \( \dot{\omega} \), so that integration can be performed in elliptic quadratures.

Let us mention another case when Eqs. (0.6) are Hamiltonian: the operator \( I \) is spherical (all its eigenvalues are the same), while operator
A is arbitrary. The integrability of this problem was proved in [5], while it was shown in [9] that Eqs. (0.6) are reducible by a linear replacement of variables \( \tilde{\omega} \) and \( e \) to Kirchhoff’s equations in the case of Clebsch integrability. The question of whether system (0.6) is Hamiltonian in the general case still remains open.

It can prove difficult to "recognize" if a concrete dynamic system is Hamiltonian. A relevant unsolved problem is the following. We are speaking of Chaplygin’s problem on the rolling of a dynamically asymmetric balanced sphere along a horizontal rough plane [10]. The equations of rolling are described by the system of differential equations

\[
\dot{k} + \tilde{\omega} \times k = 0, \quad \dot{e} + \tilde{\omega} \times e = 0; \quad k = I \tilde{\omega} + ma^2e \times (\tilde{\omega} \times e);
\]

(0.7)

here, \( \tilde{\omega} \) is the angular velocity of the sphere, \( e \) is the unit vertical vector, \( I \) is the inertia operator, \( m \) is the mass of the sphere and \( a \) is its radius. Formally, with \( a = 0 \), this problem is the same as Euler’s problem on the inertial rotation of a body.

A non-holonomic Chaplygin system appears not to be Hamiltonian. The statement of the reducibility problem can be extended, however, by additionally including a change of time along the trajectories. This method of reducing a non-holonomic system to the Hamiltonian form was used by Chaplygin in his theory of the reducing factor [10].

We need to examine if system (0.7) is Hamiltonian in the four-dimensional integral manifolds \( M_c = \{ \tilde{\omega}, e: (I \tilde{\omega}, e) = c, (e, e) = 1 \} \) (cf. [7]).

Using Chaplygin's reducing factor method, it can be shown that, after the change of time

\[
d\tau = Nd\tau, \quad N = [(ma^2)^{-1} - (e, (1 + ma^2E)^{-1}e)]^{-1/2},
\]

(0.7) reduces to a Hamiltonian system on the integral manifold \( M_0 \). Here, \( M_0 \) is diffeomorphic to the space of cotangent fibering \( T^*S^2 \), while the simplectic structure in \( M_0 \) is the standard structure in a cotangent fibering, and the Hamilton function is the kinetic energy of the sphere \( T = (k, \tilde{\omega})/2 \), referred to new time \( \tau \). The solutions of system (0.7) corresponding to the value \( c = 0 \) are the extremals of the variational problem.
\[ \delta \int_{t_i}^{t_f} T \, dt = 0, \quad \int_{t_i}^{t_f} N \, dt = \text{const.} \]

If \( c \neq 0 \), gyroscopic forces appear in the equations of motion of the reduced system, and Chaplygin's method cannot be used for this case.

2. Dynamic systems (specially Hamiltonian) are conventionally divided into integrable and non-integrable, where different definitions of integrability are possible, each with its own theoretical interest. A system which is integrable in the sense of one definition may not be integrable in another. Examples will be given below. The concept of integrability is usually linked with the presence of a fairly large number of independent integrals — "conservation laws." For instance, for the complete integrability of the Hamilton equations with \( n \) degrees of freedom (in \( M^{2n} \)) it suffices to know \( n \) independent integrals \( F_1, \ldots, F_n \), which are pairwise in involution: the Poisson brackets \( \{ F_i, F_j \} = \omega(v_{F_i}, v_{F_j}) \) are zero. It is well known that the compact level surfaces of the energy integral \( \{ H = h \} \) of a completely integrable Hamiltonian system stratify into multidimensional tori with quasi-periodic motions.

Given a non-autonomous Hamiltonian system with Hamiltonian \( H: M^{2n} \times \mathbb{R}_t \rightarrow \mathbb{R}_t \), for its complete integrability it suffices to have \( n \) independent integrals \( F_s: M^{2n} \times \mathbb{R}_t \rightarrow \mathbb{R} (s = 1, \ldots, n) \), which are in involution for all values of \( t \). The most important case for applications is that when the Hamiltonian \( H \) and the integrals \( F_1, \ldots, F_n \) are periodic in \( t \) with the same period \( p \). Then, as the extended phase space it is natural to take \( M^{2n} \times \mathbb{T}^1 \{ t \mod p \} \), and not \( M^{2n} \times \mathbb{R} \). If the integral surfaces \( \{(z, t) \in M \times \mathbb{T}^1: F_s(z, t) = c_s, 1 \leq s \leq n \} \) are compact, they are \((n + 1)\)-dimensional tori with quasi-periodic motions.

In the light of the theorem on the straightening of trajectories, the integrability of a dynamic system is best studied either in the neighbourhood of an equilibrium position, or in a sufficiently large domain of the phase space, where the trajectories are capable of being reversed.

Before studying the integrability of concrete systems, we need to refine the concept of a set of independent integrals. We shall deal exclusively with analytic Hamiltonian systems. In this situation it is very natural to consider sets of analytic integrals which are independent at least one point (they are then independent almost everywhere). It must be borne in mind, however, that an analytic Hamiltonian system may
have integrals of class $C^r$, but not integrals of class $C^{r+1}$ (we do not exclude the value $r = 0$: a continuous function will be called an integral if it is not constant locally and takes constant values on any trajectory). Suitable examples are given in [1] (Chapter V, Section 4). Let us emphasize that we are speaking here of the existence of integrals of finite smoothness, defined in the entire phase space; any analytic system of differential equations always has locally a complete set of independent analytic first integrals.

In problems of classical mechanics the phase space $\mathbb{M}^{2n}$ is usually a space of cotangent bundle $T^*\mathbb{N}^n$ ($\mathbb{N}$ is called the space of positions or the configuration space). As the canonical variables $x, y$ in $T^*\mathbb{N}$ we can take local coordinates $y_1, \ldots, y_n$ in the manifold $\mathbb{N}$ and Cartesian coordinates $x_1, \ldots, x_n$ in the $n$-dimensional layers $T^*_y\mathbb{N}$, which are the components of 1-forms in $\mathbb{N}$ in the $y$ coordinates. Hamilton functions usually depend quadratically on the momenta $x_1, \ldots, x_n$. In this situation it is natural to consider the existence of first integrals in the form of polynomials in the variables $x$ with smooth and one-valued coefficients in the space of positions $\mathbb{N}$. A good reason for this is the fact that the vast majority of known integrals in problems of dynamics have precisely the polynomial form.

3. Once mathematicians had realized the impossibility of solving the equations of classical dynamics in the closed form, strict results regarding their non-integrability began to appear. The first seems to have been Liouville's theorem (1841) concerning the unsolvability in quadratures of the equation $\ddot{x} + tx = 0$, see [11]. Then in 1887 appeared Bruns' theorem on the non-existence of algebraic integrals apart from the classical in the three-body problem [12]. This theorem was extended by Painlevé to the case when the integrals are algebraic in the velocities of three gravitating bodies. However, these classical results were of no value for dynamics, since they take no account of the peculiarities of the behaviour of the phase trajectories. It can happen that the equations of motion are completely integrable, yet they have no integrals which are say polynomial in the velocities. A simple example [13] is: the motion of a charged particle over the "flat" torus $T^2 = \{ x, y \mod 2\pi \}$ in a constant magnetic field is described by the equations

$$\ddot{x} + \nu \dot{y} = 0, \quad \dot{y} - \nu \dot{x} = 0; \quad \nu = \text{const.} \tag{0.8}$$

They have the energy integral $\dot{x}^2 + \dot{y}^2 = 2\nu$. It can be shown that Eqs. (0.8) have no auxiliary integral which is polynomial in the veloci-
ties \( \dot{x} \) and \( \dot{y} \) and has coefficients which are smooth and one-valued in \( T^2 \). However, system (0.8) is completely integrable: an auxiliary integral is e.g., the function \( \sin (\dot{x}/\nu + y) \). While the integral \( \dot{x} + \nu y \) is linear in the velocity, it is a many-valued function in the phase space \( R^2 \times T^2 \).

The statement of the problem of the non-integrability of the Hamilton equations on the whole and the first results on this subject are due to Poincaré [14]. He studies the Hamilton differential equations in the form

\[
\dot{x}_s = -\frac{\partial H}{\partial y_s}, \quad \dot{y}_s = \frac{\partial H}{\partial x_s}; \quad 1 \leq s \leq n
\] (0.9)

\[H = H_0(x_1, \ldots, x_n) + \epsilon H_1(x_1, \ldots, x_n, y_1, \ldots, y_n) + \ldots \]

The Hamiltonian \( H \) is a power series in \( \epsilon \), and its coefficients are analytic functions in \( R^n \times T^n = \{x; \ y \ \text{mod} \ 2\pi\} \). With \( \epsilon = 0 \) we have a completely integrable system. The differential equations (0.9) are often encountered in applications, so that Poincaré referred to their study as the "basic problem of dynamics." He investigated the existence for Eqs. (0.9) of the first integrals \( F(x, y, \epsilon) \), analytic in \( D \times T^n \times (-\epsilon_0, \epsilon_0) \), where \( D \) is a domain in \( R^n = \{x\} \). We showed in [1] that it is better to consider the existence of formal integrals in the form of power series \( \Sigma F_s(x, y)\epsilon^s \) with coefficients, analytic in the domain \( D \times T^n \). This problem is closely linked with the possibility of realizing the classical scheme of perturbation theory.

A more difficult problem is the existence of analytic integrals for system (0.9) with fixed values of the parameter \( \epsilon \neq 0 \). A popular problem of this type is to study the complete integrability of the Hamiltonian system close to a position of stable equilibrium. The formal analysis of this problem goes back to Birkhoff [3], while strict proofs were given by Siegel [15] (for a discussion of these topics see [16]).

In the vast majority of integrated problems of classical mechanics the known first integrals can be continued into the complex domain of variation of the phase variables as one-valued holomorphic (or meromorphic) functions. In this connection there arises the interesting problem of the complete "complex" integrability of a holomorphic Hamiltonian system. It has to be borne in mind here that the absence of holomorphic integrals in the complex domain does not necessarily imply that the Hamiltonian system is non-integrable in the real
sense. To quote a simple example: we have the linear Hamiltonian system

\[ \ddot{z} + (\mu^2 + \epsilon f(t))z = 0 \]  

(0.10)

where \( \mu, \epsilon \) are real parameters, \( f(t) \) is an elliptic function with periods \( 2\pi \) and \( 2\pi i \), which takes real values for real \( t \) and has a unique second order pole in the rectangle of periods. In the real domain system (0.10) has the analytic integral \( f(\dot{z}, z, t) \), periodic in \( t \). This fact is a consequence of the familiar result concerning the reducibility of a linear Hamiltonian system with periodic coefficients to a linear Hamiltonian system with constant coefficients by means of a suitable linear canonical transformation, periodic in \( t \). Yet for almost all \( \mu \) and \( \epsilon \) system (0.10) has no one-valued holomorphic integral in the corresponding complex phase space. This was proved in [1] (Chapter VII, Section 2) with the aid of Ziglin’s results [17].

The branching of the solutions of the Hamilton equations as functions of complex time prevents the existence of one-valued first integrals. For, since the first integral is constant along the solution, then, in the case of branching, it must take the same value at different points of the phase space. This imposes restrictions on the structure of the one-valued integral. The more complex the structure of the branching solution, the more “massive” the set in which the integral takes constant values. It can turn out on the whole that the integral reduces to a constant value in the entire phase space or in the set of simultaneous levels of the known one-valued integrals. Of course, not every branching is “dangerous.” A trivial example is the system with one degree of freedom with one-valued algebraic Hamiltonian

\[ H = \frac{x^2}{2} + f_n(y), \]

where \( f_n \) is a polynomial of degree \( n \geq 5 \) with simple roots, the general solution of which is a many-valued function of \( t \). To analyze such questions, we require information about the structure of the branching solutions. At the present time there are two constructive methods for detecting branching of the solutions: Poincaré’s method, linked with expansion of the solutions into convergent series in powers of a small parameter, and Lyapunov’s method, based on a study of linear equations in variations.
The branching properties of the solutions of Hamiltonian systems, which prevent the existence of one-valued holomorphic first integrals, were first established in [18] in terms of the small parameter method. This result was extended in [17] to non-Hamiltonian systems. A similar method was used in [19] to study the effect of self-intersection of complex separatrices in preventing the existence of holomorphic integrals. Finally, in [17] was studied the connection between the structure of the monodromy groups of the equations in variations of the solutions of Hamiltonian systems with the presence of holomorphic (and even meromorphic) integrals in the complex phase space.

In complex completely integrable Hamiltonian systems the level surfaces of involutive integrals often turn out to be, not simply the real tori \( T^n \), but, given a suitable continuation into the complex domain, the Abelian manifolds \( T^{2n} \). The general solution is then expressible in \( \vartheta \)-functions of complex time. Systems with these properties are often called "algebraically integrable." Discovery of the necessary conditions for algebraic integrability is usually based on the method employed by Kovalevski in 1888 in rigid body dynamics [20]. The present state of this subject can be learnt from van Moerbeke's paper [21] to the international congress of mathematicians in Warsaw, and from the book [22].

4. Poincaré's ideas have recently received further development, and new effects have been found in the behaviour of Hamiltonian systems that prevent their integrability. This has allowed non-integrability to be strictly proved for several important problems of Hamiltonian mechanics (heavy asymmetric top, rigid body in ideal fluid, problem of four point vortices, etc.). The mechanism of non-integrability is as follows. In the phase space of the undisturbed completely integrable system there are infinitely many resonance tori, filled by periodic trajectories: when a perturbation is added, these tori break down. From the families of periodic solutions located in them are born pairs of non-degenerate periodic solutions. On the trajectories of non-degenerate periodic solutions, the first integrals are dependent. The disrupted resonance tori accumulate, usually into pairs of twinned separatrices (asymptotic surfaces) of the undisturbed problem. On the addition of perturbation the separatrices themselves split up, and as a rule intersect, thereby forming a tangled network. The non-degenerate periodic solutions of the disturbed problem, continued into the plane of complex time, are many-valued functions, and their branching prevents the exis-
tence of holomorphic first integrals in the complex phase space. All the necessary details can be found in the author's survey [1], which summarizes the advances in this field up to 1983. The new problems that have arisen in connection with the analysis of non-integrability effects will be discussed below.

1. INTEGRABILITY CONDITIONS OF POINCARÉ’S "BASIC PROBLEM OF DYNAMICS"

1.1. Perturbation theory

Following Poincaré, we consider the "basic problem of dynamics", connected with the study of the following Hamiltonian systems:

\[ \begin{align*}
\dot{x}_s &= -\frac{\partial H}{\partial y_s}, \quad \dot{y}_s = \frac{\partial H}{\partial x_s}; \quad 1 \leq s \leq n, \\
H &= H_0(x) + \epsilon H_1(y).
\end{align*} \tag{1.1} \]

The function \( H_0 \) is a non-degenerate quadratic form in the momenta \( x_1, \ldots, x_n \) with constant coefficients:

\[ H_0 = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij} x_i x_j, \quad \det |a_{ij}| \neq 0. \tag{1.2} \]

The function \( H_1 \) is analytic in the \( n \)-dimensional torus \( T^n = \{ y_1, \ldots, y_n \mod 2\pi \} \). The Hamiltonian system (1.1) is a particular case of system (0.9) which we discussed in the introduction. While it preserves the most important features of the general case, it is simpler to analyze from the technical point of view.

If the form \( H_0 \) is positive definite, Eqs. (1.1) describe the dynamics of a natural mechanical system in the \( n \)-dimensional torus \( T^n = \{ y \} \) with kinetic energy \( H_0 \) and a small potential \( \epsilon H_1 \).

We introduce some notation. Let \( \xi = (\xi_1, \ldots, \xi_n) \) and \( \eta = (\eta_1, \ldots, \eta_n) \) be vectors in \( \mathbb{R}^n \). We put

\[ (\xi, \eta) = \sum_{i=1}^{n} \xi_i \eta_i, \quad \langle \xi, \eta \rangle = \sum_{i,j=1}^{n} a_{ij} \xi_i \eta_j. \]

The relation (1.2) can be written more briefly as
\[ H_0 = \frac{1}{2} \langle x, x \rangle. \]

An important role will be played in our future analysis by the Fourier expansion of the perturbing function:

\[ H_1 = \sum_{m \in \mathbb{Z}^n} h_m e^{i(m, y)}, \quad h_m = \text{const.} \] (1.3)

Following Poincaré, we pose the question of system (1.1) having a complete set of \( n \) independent integrals in the form of power series \( \Sigma F_k(x, y) \epsilon^k \) with coefficients that are analytic and \( 2\pi \)-periodic in \( y \). Let us emphasize that there is no reason at all to demand that these power series be convergent.

Some explanations are needed concerning the formal integrals. The formal series \( \Sigma f_s \epsilon^s \) will be regarded as equal to zero if all the \( f_s = 0 \).

The series \( F = \sum_{k \geq 0} F_k \epsilon^k \) is the formal integral of the Hamilton canonical equations with Hamiltonian \( H = \sum_{m \geq 0} H_m \epsilon^m \) if the formal series

\[ \{F, H\} = \sum_{s \geq 0} \left( \sum_{k+m=s} \{F_k, H_m\} \right) \epsilon^s \]

is equal to zero; here, \( \{,\} \), denotes the standard Poisson's brackets. The formal series are regarded as dependent when all the minors of maximum order of their Jacobi matrices vanish (as formal series in powers of \( \epsilon \)).

By using the non-degeneracy of the unperturbed problem with Hamiltonian \( H_0 \), it can be shown that the formal integrals that form the complete set of integrals are pairwise in involution.

The question of the existence of a complete set of formal independent integrals turns out to be closely linked with the possibility of realizing the classical scheme of perturbation theory. Let us recall its basic idea and mention some explicit expressions which will be used later (the details can be found in [14], Chapter IX, and in [23]). We try to find a canonical transformation \( x, y \mod 2\pi \rightarrow u, v \mod 2\pi \), dependent on parameter \( \epsilon \), such that, in the new variables, the Hamiltonian \( H_0 + \epsilon H_1 \) has the form

\[ K_0(u) + \epsilon K_1(u) + \epsilon^2 K_2(u) + \ldots \]
If such a transformation can be found, the initial system of Hamilton equations will be integrable. We shall seek the canonical transformation in the form

\[ x_i = \frac{\partial S}{\partial y_i}, \quad v_i = \frac{\partial S}{\partial u_i}; \quad i = 1, \ldots, n \]

\[ S = S_0(u, y) + \epsilon S_1(u, y) + \ldots \]

We usually put \( S_0 = \Sigma u_i y_i \); then, with \( \epsilon = 0 \), we have the identity transformation. The generating function \( S \) must satisfy the Hamilton-Jacobi equation

\[ H_0 \left( \frac{\partial S}{\partial y} \right) + \epsilon H_1(y) = K_0(u) + \epsilon K_1(u) + \ldots \]

Expanding the left-hand side in powers of \( \epsilon \) and equating coefficients of like powers of \( \epsilon \), we obtain an infinite chain of equations for finding successively \( S_1, S_2, \ldots \):

\[ \sum_k \frac{\partial H_0}{\partial u_k} \frac{\partial S_1}{\partial y_k} + H_1(y) = K_1(u) \]

\[ \sum_k \frac{\partial H_0}{\partial u_k} \frac{\partial S_2}{\partial y_k} + \frac{1}{2} \sum_{i,j} a_{ij} \frac{\partial S_1}{\partial y_i} \frac{\partial S_1}{\partial y_j} = K_2(u) \quad (1.4) \]

\[ \sum_k \frac{\partial H_0}{\partial u_k} \frac{\partial S_m}{\partial y_k} + \frac{1}{2} \sum_{i,j} a_{ij} \sum_{p+q=m} \frac{\partial S_p}{\partial y_i} \frac{\partial S_q}{\partial y_j} = K_m(u). \]

It is easily shown that Eqs. (1.4) have a unique (formal) solution \( S_1, S_2, \ldots \), in the form of trigonometric series in \( y_1, \ldots, y_n \) with zero unattached coefficients:

\[ S_m = \sum_{\tau \in \mathbb{Z}^n} S_m^\tau(u) e^{i(\tau, y)}. \]

Take the first equation of system (1.4) and let us solve it by Fourier's method. Using (1.3), we obtain

\[ K_1 = h_0; \quad S^\tau_i = \frac{i \hbar}{(\omega, \tau)}, \quad \tau \neq 0, \quad (1.5) \]

where \( \omega = (\omega_1, \ldots, \omega_n), \omega_s = \partial H_0/\partial u_s \). Clearly, \( (\omega, \tau) = \langle u, \tau \rangle \). It is evident from (1.5) that the function \( S_1 \) is not defined at points of \( \mathbb{R}^n = \{u\} \) which lie on the hyperplanes.
\[ \langle \tau, u \rangle = 0, \ h_r \neq 0. \]

The set of all these points is called the first order secular set and denoted by \( B_1 \).

On solving the second equation of system (1.4) by Fourier's method, we formally obtain \( S_2^k = i \hbar_k / \langle u, k \rangle \), where
\[
\hbar_k = \frac{1}{2} \sum_{\tau + \sigma = k} \frac{\langle \tau, \sigma \rangle h_r h_\sigma}{\langle u, \tau \rangle \langle u, \sigma \rangle}, \quad k \neq 0. \tag{1.6}
\]
The other equations of system (1.4) are solved in the same way.

1.2. Theorems on non-integrability

**Theorem 1.** If the secular set \( B_1 \) consists of an infinite number of distinct hyperplanes, then system (1.1) does not have \( n \) formal integrals \( \Sigma F_k \epsilon^k \) with analytic coefficients \( F_k : \mathbb{R}^n \times T^n \rightarrow \mathbb{R} \), which are independent for \( \epsilon = 0 \).

The theorem is a simple consequence of the following:

**Lemma (Poincaré).** Assume that system (1.1) has \( n \) integrals
\[ F_0^{(1)}(x, y) + \epsilon F_1^{(1)}(x, y) + \ldots + F_0^{(n)}(x, y) + \epsilon F_0^{(n)}(x, y) + \ldots \]
Then,

i) the functions \( F_0^{(1)}, \ldots, F_0^{(n)} \) are independent of \( y_1, \ldots, y_n \),

ii) the Jacobian \( \frac{\partial(F_0^{(1)}, \ldots, F_0^{(n)})}{\partial(x_1, \ldots, x_n)} = 0 \) at points of set \( B_1 \).

If, in the torus \( \{ x = x^0, y \mod 2\pi \} \), the frequencies \( \omega_1, \ldots, \omega_n \) are rationally incommensurable (the torus is non-resonant), then, by Weyl's uniform distribution theorem, the functions \( F_0^{(\sigma)}(x, y) \) are independent of \( y \). To prove claim i), it remains to use the density of the set of non-resonant tori in the phase space of a non-degenerate completely integrable system. A proof of claim ii) by Poincaré can be found in [14] (Chapter V, p. 82). Poincaré's original proof, see [24], was based on analysis of the birth of isolated long-periodic solutions of the disturbed problem. We shall return to considering this effect in Section 5.

We now prove Theorem 1. For this, we note the following: the union of an infinite number of distinct hyperplanes in \( \mathbb{R}^n \) which pass through
the origin, is a uniqueness set for the class of functions, analytic in \( \mathbb{R}^n \). It remains to observe that the Jacobian of functions \( F_0^{(0)}, \ldots, F_0^{(n)} \), is analytic.

In the conditions of Theorem 1, we can assert the absence of a complete set of independent integrals of the form \( \sum F_k(x, y) \epsilon^k \), whose coefficients \( F_k \) are analytic in the direct product \( D \times \mathbb{T}^n \), where \( D \) is any open domain in \( \mathbb{R}^n = \{ x \} \) which has a non-empty intersection with \( \mathbb{B}_1 \setminus \mathbb{B}_1 \).

**Theorem 1.** Let the conditions of Theorem 1 hold. If the Hamiltonian system \( (1.1) \) has \( n = 1 \) analytic integrals

\[
F^{(i)} = F_0^{(i)} + \epsilon F_1^{(i)} + \ldots + F_0^{(n-1)} + \epsilon F_1^{(n-1)} + \ldots
\]

(2.1)

where the functions \( F_0^{(i)}, \ldots, F_0^{(n-1)} \) are independent at at least one point of \( (\mathbb{B}_1 \setminus \mathbb{B}_1) \times \mathbb{T}^n \), then the Hamilton equations have no integral, independent of the functions \( F^{(i)}, \ldots, F^{(n-1)} \), in the form of a formal series \( \sum F_\epsilon \epsilon^s \) with coefficients which are analytic and one-valued in \( \mathbb{R}^n \times \mathbb{T}^n \).

Among the integrals (2.1) there can be the Hamilton function \( H_0 + \epsilon H_1 \). Note that the unique critical point of function \( H_0 \) is the point \( x = 0 \). Hence it follows in particular that, with \( n = 2 \), Theorem 1 can be strengthened, by replacing the condition that functions \( H_0 \) and \( F_0 \) be independent by the weaker condition that the series \( H_0 + \epsilon H_1 \) and \( \sum F_\epsilon \epsilon^s \) be independent.

Assume that the condition of Theorem 1 does not hold. Then, there is a solution \( S_1 \) of the first equation of system (1.4) which is analytic in \( (\mathbb{R}^n \setminus \mathbb{B}_1) \times \mathbb{T}^n \). In particular, the functions (1.6) are defined and analytic in the domain \( \mathbb{R}^n \setminus \mathbb{B}_1 \). In this situation a second order secular set \( B_2 \) can be defined as the set of hyperplanes \( \langle k, u \rangle = 0, k \neq 0 \), on which the corresponding functions \( h_k \neq 0 \).

**Theorem 1.** If the secular set \( B_2 \) consists of an infinite number of distinct hyperplanes, then system (1.1) does not have \( n \) formal integrals \( \sum F_\epsilon \epsilon^s \) with analytic coefficients \( F_\epsilon \): \( \mathbb{R}^n \times \mathbb{T}^n \to \mathbb{R} \), which are independent for \( \epsilon = 0 \).

The proof follows the same lines as for Theorem 1. We can also prove a Theorem 1', obtained from Theorem 1; by replacing \( B_1 \) by \( B_2 \).

Notice the interesting particular case when \( h_k = 0 \) for all \( k \neq 0 \). Then, \( S_2 = 0 \). We obtain by induction from system (1.4):
$S_3 = S_4 = \ldots = 0$. In this case, Eqs. (1.1) are completely integrable. If the points of the set $m = \{m \in \mathbb{Z}^n: h_m \neq 0\}$ lie on orthogonal straight lines (in the Euclidean metric $(,)$), passing through the origin, then obviously $\hat{h}_k = 0$ for $k \neq 0$. It will be shown below that the canonical variables can be separated in this case.

The secular sets of higher orders are defined recursively: if there exist solutions $S_1, S_2, \ldots, S_{p-1}$ of the first $p - 1$ equations of system (1.4), which are analytic in $(\mathbb{R}^n \setminus B_1 \cup \ldots \cup B_{p-1}) \times T'$, then the $p$-th order secular set $B_p$ is correctly defined, and Theorems $1_p$ and $1_p'$ hold. If the disturbing function $H_1$ is a trigonometric polynomial, then each of the sets $B_p$ consists of only a finite number of distinct hyperplanes (i.e., $B_p = B_p$) so that Theorems $1_p, p = 1, 2, \ldots$, lead to no conclusions about the integrability of the Hamiltonian system (1.1). This type of situation is often encountered in analysis. For instance, there are series whose convergence or divergence cannot be established by an infinite number of logarithmic tests.

If the secular sets of all orders are defined, we can put

$$B_\infty = \bigcup_{p=1}^{\infty} B_p.$$  

It turns out that Theorems $1_\infty$ and $1_\infty'$ are valid. Admittedly, these theorems are not constructive: to check if their conditions hold, we have to perform an infinite number of steps of perturbation theory.

Note an as yet unsolved problem: assume that $B_p = B_p$ for all $p \geq 1$, but $B_\infty = B_\infty$; then is it true that $H_1$ is a trigonometric polynomial?

Let us show that the conditions of Theorems $1_\infty$ and $1_\infty'$ can be made effective for the case of trigonometric polynomials. The set $m = \{m \in \mathbb{Z}^n: h_m \neq 0\}$ will be finite. As in the general case, it is invariant under the reflection $m \mapsto -m$. We shall assume that $H_1 \neq \text{const}$. Then, $m$ will contain at least two elements.

We introduce into $\mathbb{Z}^n$ the standard lexicographic order relation, denoted henceforth by $<: \sigma < \delta$, if, for the least subscript $s$ such that $\sigma_s \neq \delta_s$ we have the inequality $\sigma_s < \delta_s$. We say that $\sigma \leq \delta$ if $\sigma < \delta$ or $\sigma = \delta$.

**Definition.** Let $\alpha$ be the greatest element of the finite set $m$, and $\beta$ the greatest element of the set $m \setminus \{\alpha\}$, linearly independent of $\alpha$. We call the vector $\alpha$ the vertex of $m$, and the vector $\beta$, the vertex of $m$ adjacent to $\alpha$. 

Leaving aside the trivial case of integrability when all the points of \( m \) lie on a single straight line, through the origin, we shall assume henceforth that an adjacent vertex always exists.

**Theorem 2.** Let \( \alpha, \beta \) be vertices of \( m \) and

\[
\ell \langle \alpha, \alpha \rangle + 2 \langle \alpha, \beta \rangle \neq 0
\]  

(2.2)

for all integers \( \ell \geq 0 \). Then the Hamiltonian system with Hamiltonian \( H_0 + \varepsilon H_1 \) does not have \( n \) formal integrals \( \sum F_s \varepsilon^s \) with one-valued and analytic in \( R^n \times T^n \) coefficients, which are independent when \( \varepsilon = 0 \). The key point in the proof of Theorem 2 is to show that the hypersurface \( \Gamma_\ell = \{ x \in R^n : \langle x, \ell \alpha + \beta \rangle = 0 \} \) belongs to the secular set \( B_{\ell + 1} \). Clearly, all the surfaces \( \Gamma_\ell \) are distinct, and as \( \ell \to \infty \) they accumulate to the limiting hypersurface \( \langle x, \alpha \rangle = 0 \). Theorem 2 was proved by Kozlov and Treshchev in [25].

**Theorem 2'.** Let the conditions of Theorem 2 hold. If the Hamiltonian system (1.1) has \( n - 1 \) analytic integrals

\[
F^{(1)} = F_0^{(1)} + \varepsilon F_1^{(1)} + \ldots, \quad F^{(n-1)} = F_0^{(n-1)} + \varepsilon F_1^{(n-1)} + \ldots,
\]

where the functions \( F_0^{(1)}, \ldots, F_0^{(n-1)} \) are independent at at least one point of \( \Gamma \times T^n \), and \( \Gamma \) is the hyperplane \( \langle x, \alpha \rangle = 0 \), then Eqs. (1.1) have no integrals independent of functions \( F^{(1)}, \ldots, F^{(n-1)} \), in the form of the formal series \( \sum F_s \varepsilon^s \) with coefficients which are one-valued and analytic in \( R^n \times T^n \).

The proof is also based on using the inclusion \( \Gamma_\ell \subset B_{\ell + 1} \).

We perform a linear transformation of the angular variables \( y' = By \), where \( B \) is an integer-valued matrix whose determinant is \( \pm 1 \). We extend this transformation up to the canonical transformation \( x, y \mapsto x', y' \), by putting \( x' = (B^T)^{-1}x \). In the new variables,

\[
H_0 = \frac{1}{2} (BAB^T x', x'), \quad H_1 = \sum h_m \varepsilon^{i(m' \cdot y)},
\]

where \( m' = (B^T)^{-1}m \), \( h_{m'} = h_m \), \( A = \| a_{ij} \| \). With the trigonometric polynomial \( H_1 \) we associate the set \( m' \), consisting of vectors \( m' \in Z^n \) such that \( h_{m'} \neq 0 \). In the general case, \( m \) and \( m' \), regarded as sets of points of the integer-valued lattice \( Z^n \), are not identical: \( m' \) is the image of \( m \) under the mapping \( m \mapsto (B^T)^{-1}m \). Hence the vertices of these sets are not the same.
This remark allows the definition of vertices of set \( m \) to be extended. Let \( \alpha' \) and \( \beta' \) be vertices of the set \( m' \). Then, the integer-valued vectors \( B^T \alpha' \) and \( B^T \beta' \) will also be regarded as vertices of \( m \).

Let \( a, b \) be vectors of \( \mathbb{Z}^n \) and \( a', b' \) their images under the mapping \( y \mapsto y' \). Clearly,

\[
\langle a', b' \rangle' = (B A B^T a', b') = (Aa, b) = \langle a, b \rangle.
\]

Consequently, inequality (2.2) can be checked in the initial variables \( x, y \).

We consider in more detail the case when the quadratic form \( H_0 \) is positive definite. Then, the scalar product \( \langle , \rangle \) specifies a Euclidean metric in \( \mathbb{R}^n \). Let \( \alpha \) and \( \beta \) be linearly independent vertices of the set \( m \). By (2.2), a necessary condition for complete integrability of the Hamiltonian system (1.1) is the inequality \( \langle \alpha, \beta \rangle \leq 0 \). This means geometrically that the angle between the vectors \( \alpha \) and \( \beta \) is not less than \( \pi/2 \). These simple remarks are the key to the following analysis.

We first define a rhomboid as a convex polyhedron. For this, we take \( d \) orthogonal straight lines intersecting at a point \( O \). On each line we take two points, equidistant from and on opposite sides of point \( O \). We call the convex hull of these 2d points a \( d \)-dimensional rhomboid. With \( d = 1 \) we have an interval, and with \( d = 2 \) a rhombus, etc. The number of \( k \)-dimensional faces of the \( d \)-dimensional rhomboid is \( 2^{k+1} C_d^{k+1} \); in particular, this polyhedron has precisely 2d vertices and 2d faces. It can be shown that the \( d \)-dimensional rhomboid is the convex polyhedron, dual to the \( d \)-dimensional parallelepiped. Thus the rhomboid could also be called a co-parallelepiped.

**Theorem 3.** Assume that the Hamiltonian system (1.1) with \( n \) degrees of freedom has \( n \) independent formal integrals. Then the convex hull of the set \( m \) is a \( d \)-dimensional rhomboid, \( 1 \leq d \leq n \); on the faces of this rhomboid there are no points of \( m \) other than vertices.

We shall give the proof for the simplest, but important, case of two degrees of freedom.

Let \( a \) and \( b \) be adjacent vertices of the convex hull of \( m \). Clearly, \( a, b \in m \). Consider the integer-valued vector \( x = a - b = (x_1, x_2) \). Let \( x \) be the greatest common divisor of the integers \( x_1 \) and \( x_2 \). We put \( k = x/x = (k_1, k_2) \). Since \( k_1 \) and \( k_2 \) are relatively prime, for certain integers \( m_1 \) and \( m_2 \) we have \( m_1 k_1 + m_2 k_2 = 1 \). We take the linear transformation \( y' = By \) with matrix
\[
B = \begin{pmatrix}
-k_2 & k_1 \\
m_1 & m_2
\end{pmatrix}, \quad \det B = -1.
\]

The image of the vector \(k\) obviously has the components 0 and 1. The convex hull of the set \(m\) clearly transforms into the convex hull of the set \(m'\). In accordance with the definition, the vertex \(\alpha'\) is the vector \(a' = Ba\). Among all the points of \(m'\) located in the interval joining the points \(a'\) and \(b' = Bb\), we choose the point closest to \(a'\). This point is the vertex \(\beta'\) adjacent to \(\alpha'\).

We now use the inequality \(\langle a, \beta \rangle = \langle \alpha', \beta' \rangle \leq 0\), where \(\beta = B^{-1}\beta'\). All the more, \(\langle a, b \rangle \leq 0\). Thus the angle between vectors \(a\) and \(b\) is not less than \(\pi/2\). It is easily seen from this that the number of vertices of the convex hull of \(m\) is not greater than four. If \(b = -a\), the convex hull of \(m\) is an interval, while if \(b \neq -a\), it is a parallelogram with orthogonal diagonals, i.e., a rhombus.

It remains to show that there are no points of \(m\) other than vertices on the edge of the rhombus. For, let the point \(\beta \in m\) lie in the interval joining \(a\) and \(b\), and not be the same as point \(b\). We showed above that \(\langle \alpha, \beta \rangle \leq 0\), so that \(\langle a, b \rangle < 0\). But this contradicts the fact that vectors \(a\) and \(b\) are orthogonal. The theorem is proved.

Note. It would seem that Theorem 3 also holds in the case of a pseudo-Euclidean metric \(\langle , \rangle\). This is certainly true for \(n = 2\), as is easily seen by using arguments similar to the above for a Euclidean metric. With \(n = 2\) we have only one type of pseudo-Euclidean space. By a rhombus we understand here as usual a parallelogram with orthogonal diagonals. As the adjacent vertices approach indefinitely, the rhombus degenerates into an interval located on one of the isotropic straight lines.

By Theorems 2 and 3 we can prove:

Theorem 4 [25]. Assume that the quadratic form \(H_0\) is positive definite. Then the Hamiltonian system with Hamilton function \(H_0 + \epsilon H_1\) has a complete set of formally analytic in \(\epsilon\) first integrals, which are independent for \(\epsilon = 0\), if and only if the points of set \(m\) are located on \(d \leq n\) straight lines, which intersect orthogonally (in the metric \(\langle , \rangle\)) at the origin.

The proof of sufficiency is easy. For, let \(\ell_1, \ldots, \ell_d\) be the straight lines in \(\mathbb{R}^n\) referred to in the theorem. Denote by \(k_i \neq 0\) the point of the
set $Z^n \cap \ell_i$ closest to the origin. We augment $k_1, \ldots, k_d$ by integer-valued vectors $k_{d+1}, \ldots, k_n$ up to a basis in $\mathbb{R}^n$. We now perform a linear transformation $y' = My$ with the non-degenerate integer-valued matrix

$$
M = \begin{pmatrix}
  k_1 \\
  \vdots \\
  k_n
\end{pmatrix}
$$

and extend it up to the canonical transformation $x, y \mapsto x', y'$ by putting $x' = (M^T)^{-1}x$. In the new variables $x', y'$ the Hamilton function $H_0 + \epsilon H_1$ takes the form

$$
\frac{1}{2} \left( \sum_{i=1}^d a_{ii} x_i' + \sum_{j=1}^n \sum_{i>d} a_{ij} x_i' x_j \right) + \epsilon \sum_{i=1}^d f_i(y_i'),
$$

(2.3)

where $a_{ii}' = \text{const}$, and $f_i$ are $2\pi$-periodic trigonometric polynomials. The variables $x', y'$ can clearly be separated, and hence the Hamilton system with Hamiltonian (2.3) has the following set of $n$ independent commuting integrals

$$
F_i = \frac{1}{2} \left( a_{ii} x_i'^2 + x_i' \sum_{s>d} a_{is} x_s' \right) + \epsilon f_i(y_i'), \quad 1 \leq i \leq d
$$

$$
F_j = x_j', \quad j > d.
$$

Returning to the old variables $x, y$, we obtain a set of integrals, linear in $\epsilon$ (or quite independent of $\epsilon$), whose coefficients are analytic functions in $\mathbb{R}^n \times \mathbb{T}^n = \{ x; y \mod 2\pi \}$.

We now put $\epsilon = 1$ and consider the system with Hamiltonian $H_0 + H_1$.

**Proposition 1.** If the system with Hamiltonian $H_0 + H_1$ has $n$ integrals, polynomial in the momenta, with independent leading homogeneous forms, then the system with Hamiltonian $H_0 + \epsilon H_1$ has $n$ first integrals, analytic in $\epsilon$, which are independent for $\epsilon = 0$.

For the proof, we use the change of variables

$$
x \mapsto x/\sqrt{\epsilon}, \quad y \mapsto y, \quad t \mapsto \sqrt{\epsilon} t.
$$

(2.4)

Then, Eqs. (1.1) become Hamilton equations with Hamiltonian $H_0 + \epsilon H_1$, while a polynomial integral becomes (up to an unimportant
constant factor) equal to \( F + \sqrt{\epsilon} \Phi \), where functions \( F \) and \( \Phi \) are analytic on \( \epsilon \). Clearly, \( F \) and \( \Phi \) are integrals of the system with Hamiltonian \( H_0 + \epsilon H_1 \), one of the unattached terms \( F_0 \) or \( \Phi_0 \) being the same as the leading homogeneous form of the initial polynomial integral, which it was required to prove.

It was pointed out by S.V. Bolotin that the converse is also true. For, assume that the series

\[
\Sigma F_\epsilon(x, y)\epsilon^s, \quad F_\epsilon: \mathbb{R}^n \times T^n \to \mathbb{R}
\]

is an integral of the Hamilton system with Hamiltonian \( H_0 + \epsilon H_1 \). A change of variables, the reverse of (2.4), reduces this system to a system with Hamiltonian \( H_0 + H_1 \). The integral (2.5) then becomes the function

\[
\Sigma F_\epsilon(\sqrt{\epsilon}x, y)\epsilon^s = \Sigma \Phi_m(x, y)(\sqrt{\epsilon})^m,
\]

where \( \Phi_m \) are polynomials in the momenta with coefficients, periodic in \( y \). Since the resulting system does not contain the parameter \( \epsilon \), the polynomials \( \Phi_m \) are integrals of it. In the case of two degrees of freedom, it now follows at once that there is a polynomial integral, independent of \( H_0 + H_1 \). The question of the existence of integrals with independent leading forms in the multidimensional case requires further study. Only Proposition 1 will be used below. Notice that, if the disturbing function \( H_1 \) depends on the momenta \( x \), Proposition 1 is false.

The following curious theorem can be obtained from Theorem 4 along with Proposition 1.

**Theorem 5.** If the Hamilton system with Hamiltonian \( H_0 + H_1 \) has \( n \) polynomial integrals with independent leading forms, then it has \( n \) independent commuting polynomial integrals of not higher than the second degree.

**Note.** It seems that the theorem also holds in the more general case when the potential energy \( H_1 \) is any analytic function in \( T^n = \{ y \text{ mod } 2\pi \} \) (and not merely a trigonometric polynomial). Earlier, Bolotin and Byalyi proved Theorem 5 in the special case when \( n = 2 \) and the Hamilton system has a supplementary polynomial integral of not higher than the fourth order. Notice that the problem of a supplementary polynomial integral of a given degree is much simpler than the problem of the existence of an integral in the form of a polynomial whose degree is not fixed in advance.
1.3. Some generalizations and applications

1. The condition that the quadratic form \( H_0 = (Ax, x)/2 \) be non-degenerate can be replaced by the weaker conditions:
   i) \( Am \neq 0 \) for all integer-valued vectors \( m \neq 0 \),
   ii) the vectors \( A\alpha \) and \( A\beta \) are linearly independent.

Note that, if \( \det A = 0 \), conditions i)–ii) can only be satisfied simultaneously if \( n \geq 3 \).

We give a simple example. If

\[
A = \begin{pmatrix} 1 & \sqrt{2} & \sqrt{4} \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \alpha = (1, 0, 0)^T, \quad \beta = (1, -1, 0)^T.
\]

then matrix \( A \) is degenerate, but conditions i) and ii) are satisfied.

2. Note that Theorems 2 and 2' do not hold when the Fourier coefficients of the perturbing function \( H_1 \) depend on \( x \). An instructive counter-example is

\[
H = a^2x_1^2 + abx_1x_2 + b^2x_2^2 + \frac{\epsilon}{ax_1 - bx_2} (\sin y_1 - \sin y_2).
\]

The Hamilton system with this Hamiltonian can be integrated by separation of the variables: the analytic functions

\[
F_1 = a^3x_1^3 - ax_1H + \epsilon \sin y_1, \quad F_2 = b^3x_2^3 - bx_2H + \epsilon \sin y_2
\]

form a complete set of independent integrals.

In this problem, \( \alpha = (1, 0)^T, \beta = (0, 1)^T \), so that inequality (2.3) takes the form: \( b/a \neq -\ell/2 \) for all integers \( \ell \geq 0 \). The "limiting" line \( \Gamma = \{2ax_1 + bx_2 = 0\} \) is not the same as the line \( ax_1 - bx_2 = 0 \), at points of which the Hamiltonian is not defined. Yet integrability occurs for all (including irrational) values of the ratio \( b/a \).

Let \( \alpha', \alpha'', \ldots \) be elements of the set \( m \) located between vertices \( \alpha \) and \( \beta \) (under the lexicographic ordering \( < \) in \( \mathbb{Z}^n \)). Clearly, each of the vectors \( \alpha', \alpha'', \ldots \) is linearly dependent on \( \alpha \). It can be shown that Theorems 2 and 2' hold when the coefficients \( h_\alpha, h_{\alpha'}, h_{\alpha''}, \ldots, h_\beta \) are constant (the remaining Fourier coefficients kann then be non-constant analytic functions of the variables \( x_1, \ldots, x_n \)).

3. As an application of the results of Section 2, we prove:
Proposition 2. Let \( n = 2 \) and let the secular set \( B_1 \) consist of just two straight lines. The Hamilton equations then have an auxiliary formal integral if and only if these lines are orthogonal (in the metric \( \langle \cdot, \cdot \rangle \)).

Proof of sufficiency. Since \( B_1 \) consists of two lines, we have \( \tau = \lambda \tau_0 \) and \( \sigma = \mu \sigma_0 \) in (1.6), where \( \tau_0, \sigma_0 \in \mathbb{Z}^2 \); \( \lambda, \mu \) are integers. Orthogonality of the lines composing \( B_1 \) means that vectors \( \tau_0 \) and \( \sigma_0 \) are orthogonal. We show that in this case the Hamilton system can be integrated by separation of the variables. For, let

\[
\tau_0 = (\tau_1, \tau_2), \quad \sigma_0 = (\sigma_1, \sigma_2).
\]

We put

\[
Y_1 = \tau_1 y_1 + \tau_2 y_2, \quad Y_2 = \sigma_1 y_1 + \sigma_2 y_2.
\]

This homogeneous transformation of the angular coordinates can be uniquely continued up to the canonical transformation \( x, y \mapsto X, Y \). In the new variables

\[
H = \frac{1}{2} (A_{11} X_1^2 + 2A_{12} X_1 X_2 + A_{22} X_2^2) + \epsilon (f(Y_1) + g(Y_2)).
\]

where \( A_{ij} = \text{const} \), and \( f, g \) are analytic \( 2\pi \)-periodic functions. Since \( \langle \tau_0, \sigma_0 \rangle = 0 \), then \( A_{12} = 0 \). Hence the Hamilton system has two integrals, linear in \( \epsilon \),

\[
\frac{1}{2} A_{11} X_1^2 + \epsilon f(Y_1), \quad \frac{1}{2} A_{22} X_2^2 + \epsilon g(Y_2).
\]

Proof of necessity. If the disturbing function \( H_1 \) is a trigonometric polynomial, the proposition follows from Theorem 3.

Now take the case when \( H_1 \) is not a polynomial. We use Theorem 1'. Since \( \tau_0 \) and \( \sigma_0 \) are linearly independent, the numbers \( \lambda \) and \( \mu \) will be uniquely defined for fixed \( k = \lambda \tau_0 + \mu \sigma_0 \). By hypothesis, \( H_1 \) is not a polynomial; hence, among the numbers \( \lambda \) and \( \mu \), infinitely many are distinct. If \( \langle \tau_0, \sigma_0 \rangle \neq 0 \), it follows from (1.6) that \( B_2 \) consists of an infinite number of distinct lines, so that Theorem 1' can be applied. QED.

4. The proposition of Section 3 can be extended (with certain refinements) to a system with \( n > 2 \) degrees of freedom. Assume that all the
points of set \( \mathcal{m} \) are arranged on \( d \leq n \) straight lines through the origin, their directing vectors being linearly independent. We can then claim that the Hamilton system with Hamiltonian \( H_0 + \epsilon H_1 \) has \( n \) one-valued integrals, which are independent for all sufficiently small \( \epsilon \), if and only if these \( d \) lines are pairwise orthogonal (in the metric \( \langle \cdot, \cdot \rangle \)). With \( d = 1 \) the system is obviously integrable.

As an example, take the system with Hamiltonian

\[
H = \frac{1}{2} \sum_{i=1}^{n} x_i^2 + \epsilon \left[ f(y_1 - y_2) + \ldots + f(y_{n-1} - y_n) \right] \tag{3.1}
\]

where \( f \) is a real analytic 2\( \pi \)-periodic function. This system describes the dynamics of an "aperiodic" chain of \( n \) particles on a straight line. Apart from the energy integral, the equations of motion have a further integral \( x_1 + \ldots + x_n \), i.e., the total momentum of the system of interacting particles is conserved. It turns out that, if \( n > 2 \) and \( f \neq \text{const} \), then the system with Hamiltonian (3.1) does not have a complete set of independent integrals. For, in this case, \( d = n - 1 \) and the corresponding lines are defined by the vectors \((1, -1, 0, \ldots, 0)^T, \ldots, (0, \ldots, 0, 1, -1)^T\), which are not all pairwise orthogonal. If we "close" the chain, by adding to Hamiltonian (3.1) the term \( \epsilon f(y_n - y_1) \), our above proposition is no longer applicable: \( d = n \) lines are located in a hyperplane orthogonal to the vector \((1, 1, \ldots, 1)^T\). The integrability of the "periodic" chain essentially depends on the concrete form of the interaction potential \( f \). This subject will be considered in the next section.

1.4. Systems of interacting particles

1. The dynamic behaviour of \( n \) like interacting particles on a straight line is described by the Hamilton system with Hamiltonian

\[
H = \frac{1}{2} \sum_{i=1}^{n} x_i^2 + \sum_{i<j} f(y_i - y_j) \tag{4.1}
\]

where \( f(\cdot) \) is an even function (potential of paired interaction). Moser [26] and Calogero [27] showed that a system with Hamiltonian (4.1) is completely integrable if \( f \) is a Weierstrass \( \mathcal{R} \) function (for its degenerate cases \( z^{-2}, \sin^{-2}z, \sh^{-2}z \)). It was shown in [28] that this is the only case when there is a polynomial integral of third degree which is independent
of the integrals $H$ and $P = \Sigma x_i$. Adler and van Moerbeke [29] studied the problem of the algebraic integrability of the Hamilton system with Hamiltonian (4.1). More precisely, they considered the particular case when $f = \cos$. This is the classical version of the Gross-Neveu system, well known in theoretical physics. Using Kovalevski's method [20], it was shown in [29] that, with $n = 3$ and $n = 4$, for almost all initial conditions, the variables $x_i$ and $\exp(iy_i)$ are not meromorphic functions of complex time. In particular, the Gross-Neveu system is not algebraically integrable. In connection with this result, it is worth remarking again that an algebraically non-integrable system can be completely integrable in the real domain (see Introduction, Section 3).

We shall consider potentials $f$ which are analytic $2\pi$-periodic functions. An example is the system of three points on a circle, joined by elastic springs.

**Theorem 5.** If $f \neq \text{const}$ and $n > 2$, then the system with Hamiltonian (4.1) does not have a complete set of $n$ first integrals, polynomial in the momenta, with independent leading homogeneous forms.

**Note.** With $n = 3$ there is no supplementary integral in the form of a polynomial in the momenta, independent of the functions $H$ and $P$.

It is worth emphasizing again that the Weierstrass $\wp$ function has poles on the real axis.

2. Let us outline the proof of Theorem 5 in the case $n = 3$. We pass to an inertial baricentric measurement system; in this system the total momentum is zero. The passage can be realized by means of the canonical transformation

$$
Y_1 = y_1 - y_2, \quad Y_2 = y_2 - y_3, \quad Y_3 = y_1 + y_2 + y_3
$$
$$
x_1 = x_1 + x_3, \quad x_2 = -x_1 + x_2 + x_3, \quad x_3 = -x_2 + x_3.
$$

Putting $Y_3 = 0$, we perform the reduction to a system with two degrees of freedom, whose Hamiltonian is

$$
X_1^2 - X_1X_2 + X_2^2 + f(Y_1) + f(Y_2) + f(Y_1 + Y_2). \quad (4.2)
$$

We first take the case when $f$ is a trigonometric polynomial. Then, the convex hull of $m$ is a hexagon. In this case the absence of a new integral follows from Theorem 3.

Now assume that $f$ is not a polynomial. We use Theorem 1. Let
k = (m, n) ∈ Z², where m ≠ 0, n ≠ 0, m ≠ n. The vanishing condition for function h_k on the line ⟨k, Y⟩ = 0 can be written as

\[ \frac{f_m}{m} \frac{f_n}{n} = \frac{\bar{f}_n}{n} \frac{f_{m-n}}{m-n} - \frac{\bar{f}_m}{m} \frac{f_{m-n}}{m-n}. \]  

(4.3)

Here, f_s is the s-th Fourier coefficient of the function f; the bar denotes the complex conjugate. For an even function f, obviously, \( \bar{f}_s = f_s \).

Assume that \( f_\lambda ≠ 0 \) for some \( \lambda ≠ 0 \). If the secular set \( B_2 \) consists of only a finite number of distinct straight lines, then Eq. (4.3) will certainly hold for \( n = \lambda \) and all sufficiently large \( m \). We put \( f_m/m = a_m \). Then,

\[ a_{m+\lambda} = \frac{a_m a_\lambda}{a_m + a_\lambda}. \]  

(4.4)

Since f is not a polynomial, the coefficient \( a_m ≠ 0 \) for some indefinitely large \( m \). From (4.4) we find by induction that

\[ a_{m+s\lambda} = \frac{a_m a_\lambda}{sa_m + a_\lambda}. \]

Since \( a_m ≠ 0 \), then

\[ \lim_{s→∞} f_{m+s\lambda} = \lim_{s→∞} (m + s\lambda) a_{m+s\lambda} = f_\lambda ≠ 0. \]

Consequently, the function f is not analytic (in reality, it is not even summable in the interval \([0, 2π]\)). Hence, if \( f ≠ \text{const} \), the system with Hamiltonian (4.2) has no supplementary polynomial integral.

3. By using the results of Section 2, we can prove the non-integrability of some other familiar Hamilton systems. As an example, take the system with two degrees of freedom with the Hamiltonian

\[ H = \frac{1}{2} (x_1^2 + x_2^2) + \alpha [f(y_1 - y_2) + f(y_1 + y_2)] + \]  

\[ + \beta \Sigma f(y_1) + \gamma \Sigma f(2y_1). \]

Here, f is a periodic function; \( \alpha, \beta, \gamma = \text{const} \). In [30], Olshaneksky and Perelomov proved the complete integrability of multidimensional systems of this type if f is a Weierstrass \( \Re \) function (or a degenerate form of it).

We first take the case when the potential f is a trigonometric poly-
nomial. It turns out that the criterion for integrability is satisfaction of the equation

$$\alpha (\beta^2 + \gamma^2) = 0. \quad (4.5)$$

For, with $\gamma \neq 0$, the convex hull of $\mathbf{m}$ is the square, shown in Fig. 1. If $\alpha \neq 0$, the mid-points of the sides of the square are points of $\mathbf{m}$. In this case, Theorem 3 together with Proposition 1 guarantees the absence of a supplementary integral, polynomial in the momenta, with periodic coefficients. Let $\gamma = 0$, while $\alpha, \beta \neq 0$. Then, the convex hull of $\mathbf{m}$ is the same as the inner square, see Fig. 1. If $\alpha \neq 0$, the mid-points of its sides belong to $\mathbf{m}$ and hence Theorem 3 is again applicable.

![Structure of the set.](image)

In the general case, when the potential is any even analytic function, the criterion for integrability is likewise Eq. (4.5). The proof is by the method of Section 2.

1.5. **Existence of non-degenerate periodic solutions**

Poincaré found the conditions under which the Hamilton system (0.9) with small values of $\epsilon > 0$ has non-degenerate periodic solutions, close
to the periodic solutions of the unperturbed system [14]. We call a periodic solution non-degenerate if all but two of its characteristic exponents are non-zero. We know that one exponent vanishes because the system is autonomous, and another, because the system is assumed to be Hamiltonian. Recall also that the integrals that compose the complete involutive set are dependent on the trajectories of non-degenerate solutions. This remark was the basis of Poincaré's first proof that Hamilton systems of type (0.9) are non-integrable [24].

Our interest will be in the non-degenerate periodic solutions of system (1.1). If the disturbing function $H_1$ is a trigonometric polynomial, then Poincaré's classical theorem guarantees the existence of only a finite number of non-degenerate periodic solutions of system (1.1) with a given period (or energy). Following Treshchev [31], it will be shown below that, with $n = 2$, in this case also the disturbed Hamiltonian system (1.1) has infinitely many distinct non-degenerate closed trajectories.

We first state a generalized form of Poincaré's theorem. Given the Hamilton system with Hamiltonian

$$H_0(u) + \varepsilon H_1(u) + \ldots + \varepsilon^{k-1} H_{k-1}(u) + \varepsilon^k H_k(u, v) + O(\varepsilon^k)$$

$$u = (u_1, u_2) \in \mathbb{R}^2, v = (v_3, v_2) \in \mathbb{T}^2.$$  \hspace{1cm} (5.1)

Assume that, with $u = u^0$, the frequencies of the unperturbed problem

$$\omega_1 = \frac{\partial H_0}{\partial u_1}, \quad \omega_2 = \frac{\partial H_0}{\partial u_2}$$

are rationally commensurable; we shall assume that $\omega_1(u^0) \neq 0$. Then, the function $H_k(u^0, \omega_1 t, \omega_2 t + \lambda)$ is periodic in $t$ with some period $T$. We put

$$f(\lambda) = \frac{1}{T} \int_0^T H_k \, dt.$$  

Clearly, $f(\lambda)$ is $2\pi$-periodic in $\lambda$.

**Theorem 6.** Assume that the conditions hold:

i) the Hessian $\det \frac{\partial^2 H_0}{\partial u^2} \neq 0$ for $u = u^0$,

ii) for some $\lambda = \lambda_0$, the derivative $f' = 0$, while $f'' \neq 0$. 
Then, for small $\epsilon \neq 0$, there is a $T$-periodic solution of the disturbed Hamiltonian system; it is analytic in $\epsilon$ and, for $\epsilon = 0$, is the same as the periodic solution of the unperturbed system

$$u = u^0, \quad v_1 = \omega_1 t, \quad v_2 = \omega_2 t + \lambda_0;$$

its two characteristic exponents are equal to $\pm \gamma$, where

$$\gamma = \gamma_k(\sqrt{\epsilon})^k + O(\sqrt{\epsilon})^k)$$

$$\omega_1^2 \gamma_k^2 = f''(\lambda_0) \left[ \omega_1^2 \frac{\partial^2 H_0}{\partial u_2^2} - 2\omega_1 \omega_2 \frac{\partial^2 H_0}{\partial u_1 \partial u_2} + \omega_2^2 \frac{\partial^2 H_0}{\partial u_1^2} \right]_{u = u^0}. \quad (5.2)$$

The classical version of Poincaré's theorem is obtained with $k = 1$. The proof of Theorem 4 follows the standard scheme (see [14], paras. 42, 79; [32], Chapter IV).

Take the special case when $H_0 = \langle u, u \rangle / 2$ is a positive definite quadratic form with constant coefficients. We put

$$H_k = \Sigma h_r(u) e^{i(\tau \cdot \nu)}. \quad (5.3)$$

**Lemma.** If, for $u = u^0 \neq 0$, there is a unique integer-valued vector $\tau = (\tau_1, \tau_2)$, $\tau_2 > 0$ such that

1) $h_r(u^0) \neq 0,$

2) $\langle u^0, \tau \rangle = 0,$

then all the conditions of Theorem 6 hold, and the perturbed periodic solution is non-degenerate for small $\epsilon > 0$.

**Proof.** From the expansion (5.3) we see that

$$f(\lambda) = \sum_{\langle u^0, \tau \rangle = 0} h_r(u^0) e^{i\tau \cdot \lambda} = h_0 + h_r e^{i\tau_2 \lambda} + \overline{h}_r e^{-i\tau_2 \lambda}.$$ 

Since $f$ is periodic, the derivative $f' = 0$ for some two values of $\lambda$. Let us show that $f'' \neq 0$. If this is not true, then

$$h_r e^{i\tau_2 \lambda} - \overline{h}_r e^{-i\tau_2 \lambda} = h_r e^{i\tau_2 \lambda} + \overline{h}_r e^{-i\tau_2 \lambda} = 0.$$ 

Since $\exp(i\tau_2 \lambda) \neq 0$, then $h_r, \overline{h}_r = 0$. But this contradicts hypothesis 1) of the lemma. To show that the perturbed periodic solution is non-degenerate, it suffices to show that $\gamma_k \neq 0$. It is easily shown that
\[ \omega_1^2 \frac{\partial^2 H_0}{\partial u_2^2} - 2\omega_1 \omega_2 \frac{\partial^2 H_0}{\partial u_1 \partial u_2} + \omega_2^2 \frac{\partial^2 H_0}{\partial u_1^2} = \langle u, u \rangle \Delta, \]

where \( \Delta \) is the determinant of the quadratic form \( H_0 \). Since \( u^0 \neq 0 \) and the form \( H_0 \) is positive definite, then, by (5.2), \( \gamma_k \neq 0 \). The lemma is proved.

**Theorem 7.** Let \( \alpha, \beta \) be vertices of the set \( \mathcal{M} \), which satisfy the conditions of Theorem 2, and let \( x^0 \neq 0 \) be a point of \( \mathbb{R}^2 \) located on one of the lines \( \langle k \alpha + \beta, x \rangle = 0, k = 0, 1, 2, \ldots \), where the components of the integer-valued vector \( k \alpha + \beta \) are relatively prime. Then, at least two periodic solutions in the resonant torus \( x = x^0 \neq 0 \) of the unperturbed problem transform under perturbation into undegenerate periodic solutions of system (1.1) with the same period.

The idea of the proof is as follows. We perform the canonical transformation \( x, y \mapsto u, v \) in accordance with the relations

\[ x_s = \frac{\partial S}{\partial y_s}, \quad v_s = \frac{\partial S}{\partial u_s}; \quad s = 1, 2 \]

\[ S = S_0 + \epsilon S_1 + \ldots + \epsilon^{k-1} S_{k-1}, \]

where \( S_1, \ldots, S_{k-1} \) satisfy the first \( k - 1 \) equations of the infinite system (1.4). As a result of this transformation, the Hamilton function \( H_0(x) + \epsilon H_1(x) \) takes the form (5.1). It can be shown that the function \( H_k(u, v) \) satisfies the conditions of the lemma if the point \( u \neq 0 \) is on one of the lines \( \langle u, k \alpha + \beta \rangle = 0, k = 0, 1, \ldots \), where the components of the vector \( k \alpha + \beta \in \mathbb{Z}^2 \) are relatively prime.

Using Theorem 3, we find that, if the convex hull of \( \mathcal{M} \) is not a rhombus, then the system with Hamiltonian (1.1) with \( \epsilon > 0 \) has an infinite set of distinct non-degenerate solutions with the same period (or energy). Unfortunately, the domain of existence with respect to \( \epsilon \) of these solutions decreases indefinitely as \( k \to \infty \). For each fixed \( \epsilon > 0 \), therefore, we can guarantee the existence of a large, but finite, number of non-degenerate periodic solutions. This means that we cannot prove the non-integrability of system (1.1) for small fixed \( \epsilon > 0 \). However, we can prove that there is no family of first integrals, analytic in \( \epsilon \).

In the multi-dimensional case, we need to consider the problem of the bifurcations of the families of \( (n - 1) \)-dimensional tori with incommensurable frequencies, into which are stratified the standard resonant tori of the undisturbed system. It would be desirable to find a multi-dimensional analogue of Theorem 7.
1.6. **Perturbation theory for Hamilton systems with exponential interaction**

1. The methods described in Sections 1 and 2 for studying the integrability of Hamilton systems with a torus space of positions can be extended to Hamilton systems with a non-compact space of positions. Also, we can speak of Hamilton systems of the type

\[
\begin{align*}
\dot{x}_s &= -\frac{\partial H}{\partial y_s}, \quad \dot{y}_s = \frac{\partial H}{\partial x_s}; \quad s = 1, \ldots, n \\
H &= H_0 + \epsilon H_1, \quad H_0 = \frac{1}{2} \sum a_{ij} x_i x_j \\
H_1 &= \sum b_\lambda \exp(a_\lambda, y).
\end{align*}
\]  

(6.1)

Here, \(\|a_{ij}\|\) is a positive definite symmetric matrix, \(b_\lambda = \text{const}\), \(a_\lambda\) are vectors of \(\mathbb{R}^n\), and \(\epsilon\) is a small parameter. In order to discuss questions of convergence, we introduce the assumption that the sum representing \(H_1\) is finite. The system (6.1) describes the dynamics of a mechanical system with an \(n\)-dimensional space of positions \(N = \mathbb{R}^n\), kinetic energy \(H_0\), and potential of exponential interaction \(\epsilon H_1\).

Such systems are often encountered in applications. They include, for example, Toda finite chains [33], and the generalizations of them proposed by O.I. Bogoyavlenskii [34, 35]. We also encounter Eqs. (6.1) when studying certain homogeneous cosmological models in general theory of relativity [34].

Following the scheme of classical perturbation theory, we try to find the canonical transformation \(x, y \mapsto u, v\), dependent on \(\epsilon\), of the type

\[
x = \frac{\partial S}{\partial y}, \quad v = \frac{\partial S}{\partial u}; \quad S = S_0(u, y) + \epsilon S_1(u, y) + \ldots
\]

transforming the Hamiltonian \(H_0 + \epsilon H_1\) into the function \(K_0(u) + \epsilon K_1(u) + \ldots\). We put \(S_0 = \Sigma u_i y_i;\) then, with \(\epsilon = 0\), we have the identity transformation. The generating function has to satisfy the Hamilton-Jacobi equation

\[
H_0 \left( \frac{\partial S_0}{\partial y} \right) + \epsilon H_1(y) = K_0(u) + \epsilon K_1(u) + \ldots
\]

Hence we obtain an infinite chain of equations for finding successively \(S_1, S_2, \ldots\) and \(K_1, K_2, \ldots\) (cf. Eqs. (1.4)):
\[
\left( u, \frac{\partial S_1}{\partial y} \right) + H_1(y) = K_1(u)
\]
\[
\left( u, \frac{\partial S_2}{\partial y} \right) + \frac{1}{2} \left( \frac{\partial S_1}{\partial y}, \frac{\partial S_1}{\partial y} \right) = K_2(u)
\]
\[
\left( u, \frac{\partial S_m}{\partial y} \right) + \frac{1}{2} \sum_{p+q=m} \left( \frac{\partial S_p}{\partial y}, \frac{\partial S_q}{\partial y} \right) = K_m(u).
\]

As above, the angle brackets \(\langle, \rangle\) denote the scalar product given by the metric of \(H_0\).

Let us examine the solvability of the first equation of system (6.2). It can be assumed that all the \(a_\lambda \neq 0\): constant terms in the potential \(H_1\) will be referred to function \(K_1\). We shall seek the solution as a sum of exponential functions

\[ S_1 = \sum s_\lambda(u) \exp(a_\lambda, y). \]

Then, obviously,

\[ s_\lambda = - \frac{b_\lambda}{\langle u, a_\lambda \rangle}. \]

The coefficients of the expansion of \(S_1\) are not defined on the "resonant" planes \(\langle u, a_\lambda \rangle = 0\). Let \(B_1\) denote the union of all these planes. In the set \((\mathbb{R}^n \setminus B_1) \times N\), the function \(S_1\) is one-valued and analytic. The equation for \(S_2\) has the same form. On solving this equation by the same method, we arrive at the set \(B_2\) of new resonant planes; we stipulate that \(B_2\) entirely contains \(B_1\). On continuing this process indefinitely, we obtain a sequence of mutually imbedded sets

\[ B_1 \subset B_2 \subset \ldots \subset B_k \subset \ldots \]

(6.3)

We put

\[ B = \bigcup_{k=1}^{\infty} B_k. \]

The generating function \(S\) is not "defined" at points of \(B \times N \subset \mathbb{R}^n \times N\). Hence it is natural to regard the Hamilton system (6.1) as integrable or non-integrable, according to whether the sequence (6.3) stabilizes or not (i.e., \(B_m + s = B_m\) for all \(s \geq 0\)). The desirability of this definition of integrable system is discussed in Section 3. A sufficient condition for non-integrability is given below.
2. Denote by \( m \) the set of all vectors \( a_x \). We introduce into \( \mathbb{R}^n \) the standard lexicographic order relation. By analogy with Section 2, we give the following:

**Definition.** A maximum vector, call it \( \alpha \), of the set \( m \) is called a vertex of the set. Let \( \beta \) be the maximum among the vectors of \( m \) which are linearly independent of \( \alpha \). We call \( \beta \) a vertex of \( m \) adjacent to \( \alpha \).

In future we take the non-trivial case when not all the vectors of \( m \) are pairwise linearly dependent. In this case the adjacent vertex always exists.

**Theorem 8** [36]. If, for all integers \( \ell \geq 0 \),

\[
2 \langle \alpha, \beta \rangle + \ell \langle \alpha, \alpha \rangle \neq 0 \tag{6.4}
\]

then the set \( B_\ell \) contains the hyperplane \( \langle u, k\alpha + \beta \rangle = 0 \).

This theorem is similar to Theorem 2 and can be proved by the method of [25].

It should not be thought that every solution of system (6.2) has a singularity on the "resonant" planes. Take the simple example of the equation

\[
u_1 \frac{\partial S}{\partial y_1} + u_s \frac{\partial S}{\partial y_2} = e^{\gamma_i}.
\]

It has a solution \( e^{\gamma_i}/u_i \); the line \( u_1 = 0 \) is resonant. Yet we can indicate a solution without a singularity in the domain \( \mathbb{R}^2 \setminus \{0\} \):

\[
S = \begin{cases} 
\frac{1}{u_1} \left( \exp y_1 - \exp \frac{u_2 (u_2 y_1 - u_1 y_2)}{u_1^2 + u_2^2} \right); & u_1 \neq 0 \\
\frac{y_2}{u_2} \exp y_1; & u_1 = 0, \quad u_2 \neq 0.
\end{cases}
\]

It is obviously not a polynomial of exponential functions for all \( u_1^2 + u_2^2 \neq 0 \).

From the analytic point of view, the reason for the appearance of small denominators for the potentials with real exponential functions is precisely the same as in the case of a compact surface \( N \). The only difference is that the analytic assumption that the solution of system (6.2) can be written as a multiple Fourier series is generally stated geometrically as a condition for one-valuedness on \( N \).

3. The desirability of the definition of integrable system given in
Section 2 can be established by arguments going back to Poincaré ([14], Chapter V). We shall seek the first integrals of the Hamilton system as power series $\Sigma F_\lambda (x, y) \epsilon^\lambda$. In the case of a compact space of positions $N$ the coefficients $F_\lambda$ are assumed to be $2\pi$-periodic in $y_1, \ldots, y_n$. If $N = \mathbb{R}^n$ and the potential $H_1$ is a sum of real exponential functions, it is natural to seek $F_\lambda$ as the series

$$\Sigma f_\lambda (x) \exp (c_\lambda, y). \tag{6.5}$$

The first integrals have the same form in the integrated generalized Toda chains.

**Proposition 3.** Assume that the Hamiltonian system has $n$ integrals

$$F_0 + \epsilon F_1 + \ldots, \ldots, \Phi_0 + \epsilon \Phi_1 + \ldots$$

with coefficients of type (6.5). Then,

i) $F_0, \ldots, \Phi_0$ are independent of $y$,

ii) the Jacobian $\frac{\partial (F_0, \ldots, \Phi_0)}{\partial (x_1, \ldots, x_n)}$ vanishes at points of the set $B$.

If the set $B$ consists of an infinite number of distinct hyperplanes and the functions $F_0, \ldots, \Phi_0$ are analytic, then it follows from ii) that $F_0, \ldots, \Phi_0$ are dependent at all points of $\mathbb{R}^n$.

Let us prove e.g., i). The unattached coefficients $F_0, \ldots, \Phi_0$ are integrals of the unperturbed system with Hamiltonian $H_0$. On equating to zero the result of differentiating series (6.5) by virtue of the undisturbed system, we obtain the equation

$$\Sigma f_\lambda \langle x, c_\lambda \rangle \exp (c_\lambda, y) \equiv 0.$$ 

Let $c_\lambda \neq 0$. Then, $\langle x, c_\lambda \rangle \equiv 0$, whence $f_\lambda \equiv 0$. QED. We can prove ii) in the same way as the similar claim in the compact case.

We can arrive in another way at the definition of integrable Hamiltonian system (6.1). Following Birkhoff, we call a system integrable if it has a complete independent set of first integrals, polynomial in the momenta, with coefficients given by sums of exponential functions.

**Proposition 4.** Assume that the set $B$ consists of an infinite number of distinct hyperplanes. Then the Hamilton system with Hamiltonian
$H_0 + H_1$ does not have $n$ polynomial integrals with independent leading homogeneous forms.

We emphasize that the potential $H_1$ is not assumed here to be small. In the case of two degrees of freedom, we can guarantee the absence of a supplementary polynomial integral, independent of the function $H_0 + H_1$.

1.7. Conditions for integrability of generalized Toda chain

On the basis of Theorem 8 of Section 6, we first give a simple necessary condition for integrability of the Hamilton system (6.1).

Theorem 9. Assume that the system (6.1) is integrable, and let $\alpha, \beta$ be any two adjacent vertices of the convex hull of the set $m = \{a_i\}$. If $\alpha$ and $\beta$ are linearly independent, then

$$\frac{2 \langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \in -Z^+, \quad Z^+ = \{0, 1, 2, \ldots \}$$

and there are no other points of $m$ on the rib joining $\alpha$ and $\beta$.

The proof is similar to that of Theorem 3 for Hamilton systems with a torus space of positions, and is by the same method, using condition (6.4).

The assumption that vectors $\alpha$ and $\beta$ are independent is essential. This is shown by the example of a completely integrable system for which all the vectors $a_i$ are pairwise linearly dependent.

Corollary. Assume that any $n$ vectors of the set $a_1, a_2, \ldots, a_{n+1} \in \mathbb{R}^n$ are linearly independent. Then the system with Hamiltonian

$$H = \frac{1}{2} \Sigma x_i^2 + \epsilon \Sigma \exp(a_i, y)$$

is integrable if and only if

$$\frac{2 \langle a_i, a_j \rangle}{\langle a_i, a_i \rangle} \in -Z^+$$

for all $i \neq j$.

For, the convex hull of $m$ is in this case an $n$-dimensional simplex and hence any two vertices are adjacent. The inclusion (6.8) obviously
follows from (6.6). The sufficiency of condition (6.8) is proved in [29].

Condition (6.8) was first obtained by Adler and van Moerbeke [29] with the aid of Kovalevsky's criterion for the algebraic integrability of the system with Hamiltonian (6.7). The algebraic integrability of this system implies, in particular, that there is a complete set of integrals, polynomial in the variables $x_s$, $\exp y_s (1 \leq s \leq n)$, and that these variables are expressible in terms of $\theta$-functions of complex time. The above corollary clearly strengthens the result of Adler and van Moerbeke.

We now turn to considering the integrability of the generalized Toda chains introduced in [35]. Their dynamic behaviour is described by the Hamilton system with Hamiltonian

$$
H = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij} x_i x_j + \sum_{k,\ell=1}^{n+1} b_{k\ell} \exp [(a_k, y) + (a_\ell, y)].
$$

(6.9)

The vectors $a_1, \ldots, a_{n+1} \in \mathbb{R}^n$ and the real coefficients $b_{k\ell}$ satisfy the conditions:

A) for any $x \in \mathbb{R}^n$

$$
\max_k (a_k, x) > 0,
$$

B) for all $k$, the coefficients $b_{kk} > 0$.

It follows from condition A) that the vectors $a_1, \ldots, a_{n+1}$ are pairwise linearly independent. The convex hull of the set $\mathcal{M}$ is an $n$-dimensional simplex, stretched over the points $2a_1, \ldots, 2a_{n+1}$, and containing the origin. Since the rib of the simplex joining the vertices $2a_k$ and $2a_\ell$ contains the point $a_k + a_\ell$, a necessary condition for integrability is the vanishing of coefficients $b_{k\ell}$ for all $k \neq \ell$. Another condition for integrability amounts to satisfaction of relation (6.8). If both these conditions hold, then the system with Hamiltonian (6.9) is integrable, since, by a suitable canonical replacement of the variables, the function (6.9) can be reduced to the form (6.7).

O.I. Bogoyavlenskii indicated earlier a necessary condition for integrability of system (6.9) in an informal sense. It amounts to finiteness of the Coxeter group, generated by reflections with respect to the hyperplanes orthogonal to the vectors $a_1, \ldots, a_{n+1}$.

From the point of view of applications, an important role is played by chains with an infinite number of degrees of freedom. Some of them are integrable. Manakov and Flashka cited the system with Hamiltonian
\[
\frac{1}{2} \sum_{i=1}^{\infty} x_i^2 + \sum_{i=1}^{\infty} \exp(y_i - y_{i+1}).
\] (6.10)

For a discussion of these topics see e.g., [33]. It turns out that Theorem 8 also holds in the infinite-dimensional case. An additional requirement is the existence of vertices \(\alpha\) and \(\beta\) of the infinite set \(m\). For system (6.10), the vectors \(a_k\) have the form \((\ldots, 0, 1, -1, 0, \ldots)\). They are linearly independent and satisfy condition (6.8).

2. SPLITTING OF ASYMPTOTIC SURFACES

2.1. Theorems on non-integrability

Let \(M^{2n}\) be the phase space of the Hamilton system with \(n\) degrees of freedom, whose Hamiltonian \(H\) is a \(\tau\)-periodic function of time. We shall assume that function \(H\) is analytic in the direct product \(M^{2n} \times T^1\{t \mod \tau\}\). We thereby assume that the phase space \(M^{2n}\) is furnished with the structure of an analytic manifold, matched with the simplectic structure; the Poisson brackets of any two analytic functions is analytic in \(M^{2n}\). We regard the Hamilton system as completely integrable ("analytically" integrable) if it has almost everywhere \(n\) independent analytic first integrals \(F_1, \ldots, F_n\); \(M^{2n} \times T^1_r \rightarrow \mathbb{R}\), which are in involution:

\[
\dot{F}_i = \frac{\partial F_i}{\partial t} + \{F_i, H\} = 0, \quad \{F_j, F_k\} = 0; \quad i, j, k = 1, \ldots, n.
\]

Let \(\gamma^+, \gamma^- \subset M^{2n} \times T^1\) be \(\tau\)-periodic trajectories whose characteristic exponents do not lie on the imaginary axis. We shall not exclude the case when \(\gamma^+ = \gamma^-\). It is well known that there exist \((n + 1)\)-dimensional Lagrangian manifolds \(\Lambda^+, \Lambda^- \subset M^{2n} \times T^1\), filled continuously by trajectories of the Hamilton system which indefinitely approach \(\gamma^+, \gamma^-\) as \(t \rightarrow \pm \infty\) respectively. Recall that the manifold \(\Lambda\) is called Lagrangian if

\[
\int_\gamma ydx - Hdt = 0
\]

for any closed contour \(\gamma\) in \(\Lambda\) that can be contracted to a point. The surface \(\Lambda^+ (\Lambda^-)\) is called a stable (unstable) manifold for the trajectory \(\gamma^+ (\gamma^-)\). The intersection \(\Lambda^+ \cap \Lambda^-\) consists of trajectories which are
asymptotic to $\gamma^+$ as $t \to +\infty$ and at the same time asymptotic to $\gamma^-$ as $t \to -\infty$. If $\gamma^+ \neq \gamma^-$, such doubly asymptotic trajectories were called heteroclinic by Poincaré, while if $\gamma^+ = \gamma^-$ he called them homoclinic. He was the first to remark on the complicated structure of the intersection $\Lambda^+ \cap \Lambda^-$ in the case when the surfaces $\Lambda^+, \Lambda^-$ do not coincide, and indicate the connection of this problem with the non-integrability of a Hamiltonian system ([14], Chapter XXXIII).

The proof of non-integrability is based on the following idea: the first integral $F$ obviously takes constant values on every surface $\Lambda^+$ and $\Lambda^-$; consequently, if $\Lambda^+$ and $\Lambda^-$ have a non-empty intersection, then $F$ is constant on $\Lambda^+ \cap \Lambda^-$. In certain cases, by using the complicated structure of the set $\Lambda^+ \cap \Lambda^-$, it can be concluded from this that the analytic function $F$ is constant everywhere in $M \times T^1$. For instance, with $n = 1$, on the assumptions that $\Lambda^+ \neq \Lambda^-$ and $\Lambda^+ \cap \Lambda^- \neq \emptyset$, the Hamilton system has no non-constant analytic integral. On this fact is based the popular method of proving the non-integrability of Hamilton systems with "one and a half" degrees of freedom (see e.g., [1]). Under the condition that the intersection of $\Lambda^+$ and $\Lambda^-$ is transversal, a similar proof of non-integrability in the multi-dimensional case is only applicable when not all the characteristic exponents of at least one of trajectories $\gamma^\pm$ are real. We need to refer here to Devaney's paper [37], in which admittedly he takes the case of an autonomous Hamilton system with unstable equilibrium position $z$. It is assumed that the eigenvalues of the system, linearized at the point $z$, are equal to $\pm (\alpha \pm i\omega)$, $\alpha \omega \neq 0$, and that there is a transversal homoclinic trajectory $\gamma$ for the point $z$. Under these assumptions, we can prove:

**Theorem 1.** Given any transversal section $\Sigma$ of the trajectory $\gamma$, there is an invariant compact and hyperbolic set $\Gamma \subset \Sigma$, on which the mapping of the Poincaré sequence is topologically adjoint to the Bernoulli shift.

Thus the motion in $\Gamma$ is of a quasi-random type, and in particular, assuming that the Hamiltonian is analytic, the system has no auxiliary analytic integral in the neighbourhood of a homoclinic curve.

The condition $\alpha \omega \neq 0$ cannot be lifted. This is proved by the counter-example given by Devaney [38]. Take say the problem of the motion of a point with respect to the $n$-dimensional Euclidean sphere $S^n = \{ x \in \mathbb{R}^{n+1}: |x| = 1 \}$ in a field of force with quadratic potential $V = (Ax, x)/2$. This problem was integrated by Neumann in 1859. In order to obtain equilibria with homoclinic trajectories, we have to
identify opposite points of the sphere $S^n$. We shall assume that the eigenvalues of operator $A$ are distinct. As the point $z$ we take the unstable equilibrium corresponding to the maximum of potential $V$ in $S^n$. It turns out that

(i) the equilibrium $z$ is hyperbolic,

(ii) it has $2n$ distinct transversal homoclinic trajectories (on the energy surface that contains the point $z$),

(iii) the intersection of the stable and unstable manifolds $\Lambda^+(z) \cap \Lambda^-(z)$ contains an open neighbourhood of each of these homoclinic trajectories in $\Lambda^+(z)$ and $\Lambda^-(z)$,

(iv) the system has $n$ almost everywhere independent analytic integrals.

In this example, all the eigenvalues are real at the point $z$ ($\omega = 0$).

Devaney's theorem can be extended to heteroclinic trajectories. In this form it has an interesting application in celestial mechanics. The point is that, by Stromgren's hypothesis, the restricted three-body problem with certain rational mass ratios has several heteroclinic orbits, joining the Lagrange equilibrium solutions $L_4$ and $L_5$. If we could prove that they are transversal, we should obtain a new proof of the non-integrability of the three-body problem. This idea was realized by Danby [75] for the plane circular restricted three-body problem with equal masses. A very large number of homoclinic and heteroclinic orbits is discovered, which are transversal intersections of asymptotic surfaces.

As far as the author knows, no multi-dimensional analogues of Danby's theorem for periodic trajectories have been published.

For some purposes it is useful to introduce the $n$-dimensional sections of surfaces $\Lambda^+$ and $\Lambda^-$ by the $t = 0$ plane. We denote them by $\Sigma^+$ and $\Sigma^-$ and call them separatrices of the trajectories $\gamma^+$ and $\gamma^-$. The separatrices are Lagrangian manifolds in $M$ (after natural identification of $M \times \{0\}$ with $M$), invariant under mappings over the period $\tau$ of the Hamilton system. The fact that they are Lagrangian means that the restriction of the simplectic structure $\omega$ onto $\Sigma^+$ and $\Sigma^-$ is equal to zero. Clearly, $\Lambda^+$ and $\Lambda^-$ have a nonempty intersection if and only if $\Sigma^+$ and $\Sigma^-$ intersect.

Effective conditions for transversality of the intersection of asymptotic surfaces can be given for systems with the Hamiltonian

$$H_0(z) + \epsilon H_1(z, t) + o(\epsilon) \quad (1.1)$$
which is $\tau$-periodic in $t$. It is assumed that the unperturbed autonomous system has two non-degenerate hyperbolic equilibrium positions $z_0^{\pm}$, joined by doubly asymptotic trajectories $t \mapsto \hat{z}(t)$:

$$\lim_{t \to \pm \infty} \hat{z}(t) = z_0^{\pm}.$$  

To the points $z_0^{\pm}$ correspond the $\tau$-periodic solutions $z(t) = z_0^{\pm}$ of the undisturbed problem, which have no pure imaginary characteristic exponents. By the implicit function theorem, with small values of $\epsilon$, the perturbed system (1.1) has two $\tau$-periodic hyperbolic solutions $t \mapsto z_\epsilon^{\pm}(t)$, analytically dependent on $\epsilon$. We denote their asymptotic surfaces by $\Lambda_\epsilon^{\pm}$.

We assume that $\Lambda_\epsilon^+ = \Lambda_\epsilon^-$. The question is, whether the perturbed surfaces $\Lambda_\epsilon^+$ and $\Lambda_\epsilon^-$ can be said to coincide. To answer this question, Poincaré introduced the $\tau$-periodic function

$$I(\lambda) = \int_{-\infty}^{\infty} H_1(\hat{z}(t+\lambda), t) dt = \int_{-\infty}^{\infty} H_1(\hat{z}(t), t-\lambda) dt \quad (1.2)$$

and showed that, if $I(\lambda) \neq \text{const}$, then $\Lambda_\epsilon^+ \neq \Lambda_\epsilon^-$ for small $\epsilon \neq 0$ ([14], Chapter XXI). The integral (1.2) may be divergent. However, if $H_1(z_0^{+}, t) = H_1(z_0^{-}, t)$, then integral (1.2) can be assumed to be convergent, by subtracting from $H_1$ the function of time $H_1(z_0^{+}, t)$. This argument clearly achieves its purpose when $z_0^{+} = z_0^{-}$.

Incidentally, we can always avoid the questions of convergence if we rewrite the condition for splitting of the asymptotic surfaces $\Lambda_\epsilon^+$ and $\Lambda_\epsilon^-$ in the form

$$J(\lambda) = \frac{dI}{d\lambda} = \int_{-\infty}^{\infty} \{H_0, H_1\} (\hat{z}(t+\lambda), t) dt \neq 0. \quad (1.3)$$

This form of the splitting condition $\Lambda_\epsilon^{\pm}$ seems to have first appeared in Mel’nikov’s paper [39].

If $n = 1$, it was first shown by Poincaré that $\Lambda_\epsilon^+ \cap \Lambda_\epsilon^- \neq \phi$ for small $\epsilon$ in the homoclinic case. Consequently, if $n = 1$, $z_0^+ = z_0^-$, and condition (1.3) holds, then, with small $\epsilon \neq 0$, system (1.1) has no non-constant analytic integral, $\tau$-periodic in $t$. If $z_0^+ \neq z_0^-$, then condition (1.3) does not guarantee that the perturbed system is non-integrable. A simple example is:
\[ H = \frac{y^2}{2} - \frac{\cos^2 x}{2} + \epsilon \sin x. \]

Since the system is autonomous, \( H \) is the first integral of the equations of motion for all values of \( \epsilon \). Let us evaluate integral (1.3). In our case, \( x_0^\pm = \pm \pi/2, \ y_0^\pm = 0, \ \hat{x}(t) \to \pm \pi/2 \) as \( t \to \pm \infty \). Since \( \{H_0, H_1\} = y \cos x \), then

\[ \int_{-\infty}^{\infty} \{H_0, H_1\} \, dt = \int_{-\infty}^{\infty} \hat{x} \cos \hat{x} \, dt = \int_{-\infty}^{\infty} d \sin \hat{x} = 2. \]

A picture of the split separatrices is shown in Fig. 2.

![Figure 2](image)

**Figure 2** Integrable splitting of separatrices.

It is not by chance that the function \( J \) in our example is constant. It was shown by S.L. Ziglin that, if \( n = 1 \) and \( J(\lambda) \neq \text{const} \), then the perturbed Hamilton system is non-integrable for small \( \epsilon \neq 0 \) [40]. The condition that the function \( J \) is not constant can be written in the form

\[ \frac{dJ}{d\lambda} = \int_{-\infty}^{\infty} \{H_0, \{H_0, H_1\}\} (\hat{z}(t + \lambda), t) \, dt \neq 0. \quad (1.4) \]

Ziglin's theorem was extended to the multi-dimensional case by Bolotin ([41], see also [1]). He showed that condition (1.4) implies the absence of a complete set of involutive analytic integrals, if the characteristic exponents of the \( \tau \)-periodic solutions \( z_\epsilon^\pm (\cdot) \) are real and the disturbed system has a family of doubly asymptotic solutions \( t \mapsto \hat{z}_\epsilon(t) \). Let us mention a sufficient condition for the existence of a family of
doubly asymptotic solutions. Let the unperturbed system be completely integrable, and let \( F_1, \ldots, F_n \) be its involutive integrals, independent on \( \Lambda^+_0 = \Lambda^-_0 \). If

\[
\int_{-\infty}^{\infty} \{F_i, H_1\} (\tilde{z}_0(t), t) \, dt = 0 \tag{1.5}
\]

and

\[
\det \left| \int_{-\infty}^{\infty} \{F_i, \{F_j, H_1\}\} (\tilde{z}_0(t), t) \, dt \right| \neq 0 \tag{1.6}
\]

then there exists a family, analytic in \( \varepsilon \), of asymptotic solutions \( t \mapsto \tilde{z}_\varepsilon(t) \). These conditions take a particularly simple form if \( n = 1 \). Let \( \lambda_0 \) be a simple zero of the function \( J \). Then the asymptotic solution of the unperturbed problem \( t \mapsto \tilde{z}_0(t + \lambda_0) \) transforms under perturbation into a doubly asymptotic solution of the perturbed problem. For the proof, we have to substitute in (1.5)–(1.6) as the integral of the unperturbed problem the function \( H_0 \).

It can be shown [1, 41] that the condition that the doubly asymptotic solution \( t \mapsto \tilde{z}_\varepsilon(t) \) exists can be replaced by the convergence condition for the Birkhoff transformation which reduces the Hamiltonian in the neighbourhood of periodic solutions \( t \mapsto z^\pm(t) \) to a time-independent function. With \( n = 1 \) the Birkhoff transformation is convergent (Moser’s theorem [42]); in this case we obtain Ziglin’s theorem.

Let us emphasize that the condition that the characteristic exponents of the disturbed periodic solutions \( t \mapsto z^\pm(t) \) be real is essential for the proof of Bolotin’s theorem. It can be shown that this condition is difficult to verify, so that Bolotin’s result seems to be of an unconstructive kind. However, this is not the case: if all the multiplicators of the equilibrium solution \( z_0^\pm \) of the unperturbed problem are real, and none of them is multiple, then the perturbed periodic solutions \( z_t^\pm \) will obviously have the same properties.

The conditions for transversal intersection of asymptotic surfaces in autonomous Hamilton systems and the related obstacles to integrability are discussed in [1, 41].

2.2. Bifurcations of separatrices

Let \((x, y) = z\) be Cartesian coordinates in the plane \( \mathbb{R}^2 \). We furnish the plane with the standard symplectic structure \( \omega = dx \wedge dy \). Let
be a Hamilton function, $2\pi$-periodic in $t$ and analytic in the set $D \times \mathbb{T}^{1} \{ t \mod 2\pi \} \times (-\epsilon_{0}, \epsilon_{0})$ where $D$ is a domain in $\mathbb{R}^{2}$, $\epsilon_{0} > 0$.

With $\epsilon = 0$ we have an integrable Hamilton system with one degree of freedom. Assume that the unperturbed system has in domain $D$ three unstable positions of equilibrium $z_{1}$, $z_{2}$, and $z_{3}$, connected by doubly asymptotic trajectories $\Gamma_{1}$ and $\Gamma_{2}$ as shown in Fig. 3. The points $z_{1}$ and $z_{3}$ may coincide, but we require that $z_{1} \neq z_{2}$. The points $z_{1}$, $z_{2}$, $z_{3}$ are fixed points of a mapping over the period $T_{0}$ of the unperturbed system, while $\Gamma_{1}$ and $\Gamma_{2}$ are invariant curves of this mapping, filled by points which, with positive (negative) iterations of the mapping $T_{0}$, tend to the point $z_{2}$ ($z_{1}$) for the curve $\Gamma_{1}$ and to the point $z_{3}$ ($z_{2}$) for the curve $\Gamma_{2}$. With small values of the parameter $\epsilon \neq 0$, the points $z_{1}$, $z_{2}$, $z_{3}$ do not disappear, but transform into the fixed points $z_{1}(\epsilon)$, $z_{2}(\epsilon)$ and $z_{3}(\epsilon)$ of the perturbed mapping over the period $T_{\epsilon}$. In the general case, Poincaré showed that the unstable separatrix $\Gamma_{i}^{u}$ of the point $z_{1}(\epsilon)$ and the stable separatrix $\Gamma_{i}^{s}$ of the point $z_{2}(\epsilon)$ will no longer coincide for $\epsilon \neq 0$. The situation is similar for the separatrices $\Gamma_{2}^{u}$ and $\Gamma_{2}^{s}$ for the points $z_{2}(\epsilon)$ and $z_{3}(\epsilon)$ (see Fig. 4). The conditions for separatrices $\Gamma_{i}^{u}$ and $\Gamma_{i}^{s}$ to be distinct were discussed in the previous paragraph. If they are distinct, they may or may not intersect as sets of points in $\mathbb{R}^{2}$. If say $\Gamma_{i}^{u}$ and $\Gamma_{j}^{u}$ intersect, then the trajectories of the heteroclinic asymptotic solutions "interconnecting" points $z_{1}(\epsilon)$ and $z_{2}(\epsilon)$ will pass through the point of intersection. Sufficient conditions for intersection and non-intersection of separatrices with small $\epsilon \neq 0$ in the domains of $\mathbb{R}^{2}$ which do not contain the points $z_{1}$, $z_{2}$, $z_{3}$ are indicated in Section 1. The question of the fate of
the perturbed separatrices in the neighbourhood of points $z_s(\varepsilon)$ remains open. We describe below the results of S.A. Dovbysh concerning the mutual disposition of separatrices $\Gamma_1''$ and $\Gamma_2''$ in the neighbourhood of point $z_2(\varepsilon)$ [43].

We introduce some notation. Let $t \mapsto \hat{z}_1(t)$ be a doubly asymptotic solution of the undisturbed problem such that

$$\lim_{t \to -\infty} \hat{z}_1(t) = z_1, \quad \lim_{t \to +\infty} \hat{z}_1(t) = z_2.$$ 

Let $\hat{z}_2(\cdot)$ be another doubly asymptotic solution, connecting points $z_2$ and $z_3$. We put

$$J_s(\varphi) = \int_{-\infty}^{\infty} \{H_0, H_1\} (\hat{z}_s(t - \varphi), t) dt, \quad s = 1, 2.$$ 

These functions played a role in our analysis of splitting of separatrices $\Gamma'_s$ and $\Gamma''_s$ in Section 1. Note that functions $J_s$ are analytic and $2\pi$-periodic. For the case of homoclinic motions, their averages over a period are zero. However, this is by no means essential in our situation. The necessary condition for non-intersection of the disturbed separatrices $\Gamma'_s$ and $\Gamma''_s$ is that the $J_s$ undergo no changes of sign. We assume that this condition is satisfied. Moreover, we shall assume that $J_1 \geq 0$ and $J_2 \leq 0$. In this case the picture of the disposition of the split separatrices is precisely that shown in Fig. 4 in the case of small positive $\varepsilon$. With $\varepsilon = 0$, in the neighbourhood of point $z_2$, a Birkhoff canonical
transformation $x, y \mapsto \xi, \eta$ can be performed such that, in the new variables, $H_0(x, y) = F_0(\xi)$, $\xi = \xi \eta$ and

$$\frac{dF_0}{d^2 \xi} (0) = \Lambda > 0. \quad (2.1)$$

The positive eigenvalue $\Lambda$ of the undisturbed linearized system will play a role in our future discussions.

**Theorem 2.** The disturbed separatrices $\Gamma''_1$ and $\Gamma''_2$ do not coincide for small values of the parameter $\epsilon > 0$ if at least one of the following conditions holds:

(i) $\frac{d}{d\varphi} \ln J_1(\varphi) \geq \Lambda$ or $\frac{d}{d\varphi} \ln (-J_2(\varphi)) \leq -\Lambda$ for some $\varphi$,

(ii) the domains of variation of functions $J_1$ and $-J_2$ are not the same,

(iii) one of functions $J_1$ or $J_2$ is one-valued on the complex plane, while the other is not reducible to an identical constant and has a zero or pole on its Riemann surface,

(iv) $\frac{d^2 F_0}{d^2 \xi} (0) \neq 0$ and at least one of functions $J_1$ or $J_2$ is not constant.

The first condition is certainly satisfied if $J_1$ or $J_2$ vanishes for some $\varphi$. Similarly, condition (iii) is satisfied if $J_1$ and $J_2$ are trigonometric polynomials of $\varphi$ and at least one of them does not reduce to a constant. Theorem 2 is proved by means of a uniform version of Moser's theorem on the convergence of the normalizing Birkhoff transformation in the neighbourhood of the hyperbolic point $z_2(\epsilon)$ and by using the technique of [40].

These general considerations were applied by S.A. Dovbysh to the familiar problem on the rotation of an asymmetric rigid body with a fixed point in a weak homogeneous field of gravity force. The small parameter here is the product of the weight of the body with the distance from the centre of mass to the point of suspension. By factorization with respect to the group of rotations about the vertical, the problem can be reduced to a Hamiltonian system with two degrees of freedom. On further fixing the positive value of the constant energy integral and using Whittaker's isoenergy reduction method, the equa-
tions of motion are reducible to Hamilton equations with one and a half degrees of freedom with Hamiltonian of the type above considered, periodic in the new time (all the details can be found in [32]). In this problem the diagram of the separatrices of the undisturbed Euler problem (in the asymmetric case) is as shown in Fig. 5 (the points \(z_1\) and \(z_3\) coincide, since the system phase space is a cylinder and not a plane). A feature of the problem is that the characteristic numbers \(z_1\) and \(z_2\) for the hyperbolic equilibria are the same. We isolate three separatrices \(\Gamma_1, \Gamma_2, \Gamma_3\), as shown in Fig. 5.

The study of the splitting of the separatrices with small values of \(\epsilon \neq 0\) was begun in [44]. It was shown that, for almost all values of the parameters (the values of the moments of inertia of the body and the coordinates of the centre of mass in the principal axes of inertia), the perturbed separatrices do not coincide. Moreover, for a certain ratio of the problem parameters (known as the Hess-Appelroth case), one pair of separatrices remains twinned on perturbation. Ziglin refined this result by showing that the perturbed separatrices split in all cases except for one pair of separatrices under the Hess-Appelroth condition [40]. It was also found that, with certain parameter values, the split separatrices intersect, while with others there is no intersection. We have, nevertheless:

**Proposition 1.** Apart from the Hess-Appelroth condition, for all parameter values, with small \(\epsilon \neq 0\), there are at least two doubly asymptotic (homoclinic) solutions for every perturbed unstable solution \(z_1(\epsilon)\) and \(z_2(\epsilon)\). In the Hess-Appelroth case, there are no such solutions.

This claim was made orally by S.L. Ziglin. Its proof is based on simple arguments involving the application of Moser's theorem about invariant curves and the preservation of area under mappings over a period. Arguments of this kind were originally used by Poincaré for
proving the existence of homoclinic solutions when a loop of a separatrix is split. Different cases of the mutual disposition of perturbed separatrices are shown in Fig. 6 (case c) is impossible.

It remains unclear whether some of these homoclinic trajectories lie on twinned separatrices. The answer is given by Theorem 2: the integrals $J_\epsilon(\phi)$, evaluated along double asymptotic solutions, corresponding to separatrices $\Gamma_1, \Gamma_2, \Gamma_3$, are non-constant trigonometric polynomials. Hence condition (iii) of Theorem 2 can be applied.

**Theorem 3.** There exist domains $S_1, S_2, S_3$ in the space of parameters (to which must be added the constants of the energy integrals and momentum) such that

1. for parameter values of the domain $S_1 \cup S_2 \cup S_3$, the disturbed separatrices $\Gamma'_1$ and $\Gamma''_1$ split, do not intersect, and are arranged as shown in Fig. 7,

   ![Diagram](image)

   **Figure 7** Splitting of separatrices in perturbed Euler problem.

2. for parameters from the domain $S_1$, with all small $\epsilon > 0$, the disturbed separatrices $\Gamma'_1$ and $\Gamma'_3$ do not intersect close to the undisturbed separatrices $\Gamma_1, \Gamma_2, \Gamma_3$,

3. for parameters of domain $S_2$, for all small $\epsilon > 0$, the separatrices $\Gamma'_1$ and $\Gamma'_3$ intersect close to the curves $\Gamma_1, \Gamma_2, \Gamma_3$,

4. for parameters of domain $S_3$, there are positive number sequences $\epsilon^+_n \to 0$ and $\epsilon^-_n \to 0 (n \to \infty)$ such that, for $\epsilon = \epsilon^+_n$, the
separatrices \(\Gamma'_{1}\) and \(\Gamma'_{3}\) intersect close to the curves \(\Gamma_{1}, \Gamma_{2}, \Gamma_{3}\), while for 
\(\epsilon = \epsilon_{n}'\) they do not intersect.

A non-trivial effect is thus seen for points of domain \(S_{3}\): as the positive parameter \(\epsilon\) tends to zero, there occur an infinite number of bifurcations of birth and death of heteroclinic solutions passing close to the unperturbed separatrices \(\Gamma_{1}, \Gamma_{2}, \Gamma_{3}\).

Conclusion (2) of Theorem 3 admits of an interesting refinement, again due to S.A. Dovbysh.

**Proposition 2.** There is a domain \(S_{0} \subset S_{1}\) in the parameter space such that, with small \(\epsilon > 0\), there is a closed, analytic invariant curve located close to the undisturbed separatrices and dividing the disturbed fixed points \(z_{1}(\epsilon)\) and \(z_{2}(\epsilon)\).

In particular, in these cases there are no heteroclinic motions: the separatrices of the hyperbolic points \(z_{1}(\epsilon)\) and \(z_{2}(\epsilon)\) do not intersect, and remain on opposite sides of the closed invariant curve. The proof of Proposition 2 is based on Moser's theorem about invariant curves, and uses the technique developed when proving Theorem 2.

**2.3. Splitting of separatrices and birth of isolated periodic solutions**

We again consider the Hamilton equations

\[
\dot{x} = H_{1}', \quad \dot{y} = -H_{1}'; \quad (x, y) = z \in \mathbb{R}^{2}
\]

with analytic Hamiltonian \(H_{0}(z) + \epsilon H_{1}(z, t) + o(\epsilon), 2\pi\)-periodic in \(t\).

We assume that the unperturbed system with Hamiltonian \(H_{0}\) satisfies the two conditions:

(i) the point \(z = 0\) is a non-degenerate critical point of function \(H_{0}\) of index 1 (saddle point);

(ii) there is a bounded connected component \(\sigma\) of the set \(\{z: H_{0}(z) = 0\} \setminus \{z: dH_{0}(z) = 0\}\), whose closure is \(\sigma \cup \{0\}\).

Thus the point \(z = 0\) is an unstable equilibrium of the unperturbed system, while the curve \(\sigma\) is its homoclinic trajectory (separatrix loop).

Let \(T_{t}\) be a mapping over period \(t = 2\pi\) of the disturbed system. The point \(\zeta \in \mathbb{R}^{2}\) is a periodic point of \(T_{t}\) of period \(m \in \mathbb{N}\) if \((T_{t})^{m}\zeta = \zeta\).

The periodic points and only they are initial values (at \(t = 0\)) for the periodic solutions of the Hamilton system. If \(m\) is the period of point \(\zeta\), then \(2\pi m\) is the period of the solution \(t \mapsto z(t, \zeta), z(0, \zeta) = \zeta\). A
periodic point $\zeta$ is called non-degenerate if the eigenvalues of the mapping $z \mapsto (T^m_r) z$, linearized in the neighbourhood of point $\zeta$, are different from unity. Clearly, the uncritical bounded level lines of $H_0$ are composed entirely either of degenerate periodic, or of non-periodic points, of the mapping $T_0$.

We introduce the function

$$J(\lambda) = \int_{-\infty}^{\infty} \{H_0, H_1\} (\tilde{z}(t + \lambda), t) \, dt,$$

where $t \mapsto \tilde{z}(t)$ is one of the homoclinic solutions that corresponds to the separatrix $\sigma$. Since the mean over a period of function $J$ is zero, $J$ must vanish somewhere.

**Theorem 4 [45].** Assume that $J$ has a simple zero $\tilde{\lambda}$. Then, there exists an infinite number of analytic functions $\zeta_n : (-\epsilon_n, \epsilon_n) \to \mathbb{R}^2$, $\epsilon_n > 0$, such that

1. $\zeta_n(\epsilon)$ is a periodic point of the mapping $T_\epsilon$ for all $-\epsilon_n < \epsilon < \epsilon_n$, which is not degenerate for $\epsilon \neq 0$;

2. $H_0(\zeta_n(0)) < 0$ and the distance from points $\zeta_n(0)$ and $\tilde{z}(\tilde{\lambda}) \in \sigma$ tends to zero as $n \to \infty$.

In the general case the sequence of $\epsilon_n$ converges to zero. Hence, with sufficiently small $\epsilon \neq 0$, the theorem guarantees the existence of a large but finite number of distinct non-degenerate long-periodic solutions of the perturbed Hamiltonian system. Note that, if all the zeros of $J$ are simple, then the homoclinic trajectories of the perturbed system are transversal, and by the methods of symbolic dynamics, we can prove the existence of an infinite number of distinct periodic points of the mapping $T_\epsilon$ in the neighbourhood of the loop $\sigma$ (see [46]). These periodic points have, however, no connection with the points $\zeta_n(\epsilon)$.

The proof of the theorem is based on the use of Poincaré's small parameter method. For this, we transform from variables $x, y$ to simplectic variables the action and angle $I, \varphi \mod 2\pi$ of the unperturbed integrable system in the domain of $\mathbb{R}^2$ given by the inequalities $-c < H_0(z) < 0$, where $c$ is a small positive constant. Recall that

$$I = \frac{1}{2\pi} \iint_{H_0 < h} dx \wedge dy.$$
The function $h \mapsto I(h)$ is monotonic; we denote the inverse function by $F_0(I)$. In the new variables

$$H = F_0(I) + \epsilon F_1(I, \varphi, t) + O(\epsilon),$$

where $H_0(z) = F_0(I)$, $H_1(z, t) = F_1(I, \varphi, t)$. The Hamilton function is obviously $2\pi$-periodic in $\varphi$ and $t$. Let

$$\omega(I) = \frac{dF_0}{dI} -$$

be the frequency of oscillation in the unperturbed system. It is easily seen that $\omega(I) \to 0$ when the level line $H_0(z) = F_0(I)$ indefinitely approaches the separatrix loop $\sigma$. Consequently, there is an infinity of distinct values $I_n$ such that $\omega(I_n) = 1/n$. It is claimed that, on the invariant curves $I = I_n$, with small values of $\epsilon$, there occurs immediately the birth of non-degenerate periodic points $\xi_n(\epsilon)$ of the mapping $T_\epsilon$, which are spoken about in the theorem. In order to prove this, we have to check that the conditions of Poincaré's well known theorem (cf. Theorem 6 of Chapter I) are satisfied:

(a) $\frac{d^2 F_0}{dI^2}(I_n) \neq 0$,

(b) the $2\pi$-periodic function

$$f_n(\lambda) = \int_{-\pi n}^{\pi n} F_1(I_n, (t + \lambda)/n, t) dt$$

has a non-degenerate critical point $\lambda_n$.

We shall show that $d^2 F_0/dI^2 < 0$ for $I = I_n$ for sufficiently large $n$. We use the asymptotic form of function $h \mapsto I(h)$:

$$I = -\Lambda h \ell n(-h) + \ldots, \quad (3.1)$$

where the positive constant $\Lambda$ is given by Eq. (2.1), and the dots denote a power series which is convergent for small values of $h$.

Regarding $h$ as a function of $I$, we differentiate the identity (3.1):

$$1 + \Lambda h' (\ell n(-h) + \ldots) = 0. \quad (3.2)$$

Hence it follows that, with values of $I$ close to $I(0)$ (where $I(0)$ is the area enclosed inside the separatrix $\sigma$), the frequency $\omega(I) = h' < 0$. If we again differentiate (3.2), we obtain
\text{hh}''(\ell n(-h) + \ldots) + h'^2(1 + \ldots) = 0. \quad (3.3)

The dots in the second parentheses denote a power series in \( h \) without an unattached term. We conclude from (3.3) that, with \( I \) close to \( I(0) \), the second derivative \( h'' \) is negative, which it was required to prove.

Let us now analyze condition (b). We first note that the critical points of function \( f_n \) are the same as the zeros of the function

\[
f'_n = \int_{-\pi n}^{\pi n} \{H_0, H_1\} (z_n(t + \lambda), t) \, dt,
\]

where \( z_n(\cdot) \) are \( 2\pi n \)-periodic solutions of the unperturbed problem. If \( z_n(0) \to \tilde{z}(0) \) as \( n \to \infty \), the sequence \( f'_n(\lambda) \) is convergent, uniformly with respect to \( \lambda \), to \( J(\lambda) \). Using the obvious relation

\[
\frac{d^n J}{d\lambda^n} = \int_{-\infty}^{\infty} \left\{ H_0 \{ \ldots \{ H_0, H_1 \} \ldots \} \right\} \, dt,
\]

we conclude that \( f''_n \to J' \) uniformly with respect to \( \lambda \). Since, by hypothesis, the function \( J \) has a simple zero, the functions \( f_n \) with sufficiently large \( n \) will have property (b). At the same time we have shown that the sequence of points \( z_n(\lambda_n) \) is convergent to \( \tilde{z}(\tilde{\lambda}) \), where \( \tilde{\lambda} \) is the simple zero of function \( J \). The theorem is proved.

Note that the perturbed system may not have non-degenerate long-periodic solutions of period \( 2\pi/\omega = 2\pi n/m \) with \( m \neq 1 \). To be more precise, the existence of such solutions does not in general follow from a consideration of the first order disturbance with respect to \( \epsilon \). A suitable example is the familiar problem of the plane oscillations of a satellite in an elliptic orbit [47]. The transversal intersection of the separatrices in this problem for small non-zero values of the orbit eccentricity was proved in [48]. The symbolic dynamics of the quasi-random oscillations (due to Alekseev) of the satellite were studied in [49].

The above theorem can be used for proving the existence of families of non-degenerate long-periodic solutions in Hamiltonian systems whose non-integrability is proved by the method of splitting of separatrices. Some examples of such systems will be mentioned in the next section.
2.4. Some applications

We quote in this section some recent results concerned with proving the non-integrability of specific Hamiltonian systems of mechanics and mathematical physics.

1. The rotation of a heavy rigid body about a fixed point is described by the well-known Euler-Poisson equations

\[
\begin{align*}
\dot{A}p &= (B - C)qr - Mg(y\gamma_3 - z\gamma_2) \\
\dot{B}q &= (C - A)rp - Mg(z\gamma_1 - x\gamma_3) \\
\dot{C}r &= (A - B)pq - Mg(x\gamma_2 - y\gamma_1)
\end{align*}
\]

\(\gamma_1 = r\gamma_2 - pq, \quad \gamma_2 = p\gamma_3 - r\gamma_1, \quad \gamma_3 = q\gamma_1 - pr\). \hspace{1cm} (4.2)

Here, \(p, q, r\) are the projections of the angular velocity onto the principal axes of inertia; \(A, B, C\) are the moments of inertia of the body; \(M\) is its weight; \(g\) is the acceleration due to gravity; \(x, y, z\) are the coordinates of the centre of mass in the principal axes of inertia; \(\gamma_1, \gamma_2, \gamma_3\) are the direction cosines of the vertical in these axes. Equations (4.1) and (4.2) have three integrals:

(i) \(\frac{1}{2}(Ap^2 + Bq^2 + Cr^2) + Mg(x\gamma_1 + y\gamma_2 + z\gamma_3)\) is the energy integral,

(ii) \(Ap\gamma_1 + Bq\gamma_2 + Cr\gamma_3\) is the "area" integral,

(iii) \(\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1\) is a geometric relation.

It is well known that the Euler-Poisson system is Hamiltonian on the joint surfaces of the area integral and the geometric integral: the role of Hamilton function is naturally played by the total energy of the body. Thus, for complete integrability, we still lack a further integral, independent of the classical integrals (i)-(iii). It is shown in [50] and [40] that there is no new analytic integral for an asymmetric body \((A > B > C)\) with small non-zero values of the Poincaré parameter \(\epsilon = Mg\sqrt{x^2 + y^2 + z^2}\). The existence of integrals (in the real domain) for a symmetric body (when \(A = B\)) still remains an open question. In this case the method of splitting of separatrices is not applicable, in view of their absence at \(\epsilon = 0\). Below, some results of a particular kind in this subject are indicated. They are obtained in [51].

In the case when \(A = B\) it can be assumed without loss of generality that \(x = 0\). In suitable length and mass units, \(A = B = 1\). We take a rigid body in which \(C = \delta, mgy = \delta\). It turns out that, with small
values of the parameter $\delta > 0$, system (4.1), (4.2) is non-integrable.

We shall first show that such a body exists. For this, we consider three mutually perpendicular axes $X, Y, Z$, and locate on the $X$ axis, on opposite sides of the origin, at unit distance, two like masses $\delta/4$, and similarly, on the $Z$ axis, at unit distance, two like masses $(1/2 - \delta/4)$, and finally, at distance $1/2$ from the origin, we locate on $Y$ axis points with masses $\delta(1 + 1/g)$ and $\delta(1 - 1/g)$, $g > 1$. It is easily seen that all the above conditions are then satisfied.

Purely for simplicity, we take the case when $z = 0$. In the light of our assumptions, Eqs. (4.1) will have the form
\[ \dot{p} = (1 - \delta)qr - \delta \gamma_3, \quad \dot{q} = (\delta - 1)pr, \quad \dot{r} = \gamma_1. \] (4.3)
We let $\delta$ tend to zero. Then, Eqs. (4.3) become
\[ \dot{p} = qr, \quad \dot{q} = -pr, \quad \dot{r} = \gamma_1. \] (4.4)
These equations, along with (4.2), form a closed system of the restricted problem on the rotation of a heavy rigid body with a fixed point.

The significance of the restricted statement of the problem is as follows. As $\delta \to 0$, the body degenerates into a straight segment, which rotates about a fixed point according to the law of a spherical pendulum. The well known picture of the motion of this pendulum gives a clear idea of the nutation and precession of the rigid body. At first sight it seems that the problem of the natural rotation of the body becomes meaningless when $\delta = 0$. But this is not the case: as $\delta \to 0$, the moment of inertia and the moment of the gravity force relative to the axis of dynamic symmetry simultaneously tend to zero. In the limit, a non-trivial equation is obtained for the natural rotation, which we study below. Notice that the transition to the restricted problem in the dynamics of a rigid body is exactly similar to the transition to the restricted three-body problem in celestial mechanics.

In spite of our simplifications, the system (4.2), (4.4) is still non-integrable. To prove this, we first write its known integrals, obtained from integrals (i)–(iii) of the initial problem by passage to the limit:
\[ p^2 + q^2 = 2h, \quad p \gamma_1 + q \gamma_2 = c, \quad \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1. \] (4.5)
With $2h > c^2$, relations (4.5) cut out in the six-dimensional phase space of system (4.2), (4.4) a three-dimensional integral manifold $M_{h,c}$. We put
\[ p = \sqrt{2h} \sin \xi, \quad q = \sqrt{2h} \cos \xi. \]
The variables $\xi, \dot{\xi} = r$, and $\gamma_3$ are coordinates in $M_{h,c}$. It is easily shown that the coordinate $\xi$ satisfies the equation

$$\dot{\xi} = \frac{c}{\sqrt{2h}} \sin \xi - \sqrt{1 - \frac{c^2}{2h}} \sin(\sqrt{2h} t) \cos \xi. \quad (4.6)$$

It was shown in [51] that the question of the existence of an auxiliary integral of Eqs. (4.2), (4.4) reduces to the question of the existence of an integral of Eq. (4.6) which is periodic in $t$ with period $2\pi/\sqrt{2h}$. The second question can be answered in the negative by using the method of splitting of separatrices. For this, we first write (4.6) in the Hamilton form

$$\dot{\xi} = H_\eta, \quad \dot{\eta} = -H_\xi$$

$$H = \frac{\eta^2}{2} + \frac{c}{\sqrt{2h}} \cos \xi + \sqrt{1 - \frac{c^2}{2h}} \sin(\sqrt{2h} t) \sin \xi. \quad (4.7)$$

We put $\sqrt{1 - c^2/2h} = \nu$ and regard $\nu$ as a small parameter. Note that Eqs. (4.7) still have a meaning when $\nu = 0$, when degeneration of the manifold $M_{h,c}$ occurs. The Hamiltonian of system (4.7) can be written as

$$H = H_0 + \nu H_1 - o(\nu)$$

$$H_0 = \frac{\eta^2}{2} + \cos \xi, \quad H_1 = \sin \xi \sin(\sqrt{2h} t).$$

With $\nu = 0$ we have the integrable problem of a mathematical pendulum which has a separatrix loop. It was shown in [51] that this loop splits for small $\nu = 0$. Since the behaviour of the split separatrices is stable with respect to small parameter variations, then Eqs. (4.2), (4.3) have no supplementary analytic integral for small values of the parameter. Notice that, with $\delta = 1$ and $\delta = 2$, the system is integrable (the case of total dynamic symmetry and Kovalevsky's case).

Dovbysh showed by numerical computations that system (4.7) also has intersecting separatrices for the zero value of the constant area $c$. This result clearly cannot be obtained by perturbation theory. We pass to new time $\tau = \sqrt{2h} t$ and put $\xi = \pi/2 + x$. In the new variables $x$, $\tau \mod 2\pi$, system (4.7) can be rewritten as

$$\frac{dx}{d\tau} = y, \quad \frac{dy}{d\tau} = 2h \sin \tau \sin x.$$
For all values of \( h \), this system has the trivial solution, \( 2\pi \)-periodic in \( \tau \),
\[
x(\tau) = 0, \quad y(\tau) = 0.
\] (4.8)

Dovbysh established that with \( h = 1 \) and \( h = 5/2 \), the solution (4.8) is hyperbolic, and the pairs of its stable and unstable separatrices intersect, as shown in Fig. 8. It seems that transversal intersection of the separatrices occurs for all values of \( h \) for which the solution (4.8) is hyperbolic.

![Figure 8 Separatrices in restricted problem of rigid body dynamics.](image)

2. We also consider the problem of the inertial rotation of a rigid body about a fixed point with an asymmetric rotor, which can freely rotate about an axis rigidly connected with the body. This mechanical system obviously has four degrees of freedom. The space of positions is the direct product \( SO(3) \times S^1 \).

The Hamilton equations of motion have four first integrals: the total energy \( H \) is conserved, along with the three projections of the moment of momentum of the body + rotor onto the axes of a fixed orthogonal reference system, call them \( F_1, F_2, F_3 \). It is easily shown that
\[
\{F_1, F_2\} = F_3, \quad \{F_2, F_3\} = F_1, \quad \{F_3, F_1\} = F_2.
\]

Consequently the functions \( H, F_1, F^2 = F_1^2 + F_2^2 + F_3^2 \) are in involution. Thus, for complete integrability of the equations of motion, there is still lacking one independent integral, which commutes with the functions \( H, F_1, F^2 \). If e.g., the rotor is symmetric with respect to its axis of rotation, the supplementary integral is the projection of the moment of momentum of the flywheel onto its axis of rotation. This integrable case was first noted by N.E. Zhukovskii [52]; Volterra wrote the general solution of the equations of motion of a rigid body with a symmetric rotor with the aid of Weierstrass \( \sigma \)-functions [53].
In [54] E.A. Ivin used the method of splitting of separatrices to show that this problem is non-integrable in the general case. To be more precise, he considered the rotation of a rigid body with a rotor of small mass. The unperturbed problem is an integrable Euler problem on the free rotation of a rigid body, with a pair of doubly asymptotic surfaces. He showed that the asymptotic surfaces split on the addition of a rotor in a "non-integrable" way, and this implies the absence of new analytic integrals.

A similar result was obtained earlier by Marsden and Holms [55] under extra simplifying assumptions of a technical kind: in order to simplify the calculations some terms were dropped in the exact expression for the Hamilton function. In [56] an attempt was made to give a strict proof of the non-integrability of the problem on the motion of a body with an asymmetric rotor. The theoretical assumptions underlying the working of [56] were false, however.

3. For the homogeneous two-component model the Young-Mills classical fields are Hamiltonian with the Hamilton function

\[ H = \frac{1}{2} (y_1^2 + y_2^2) + \frac{x_1^2 x_2^2}{2}. \]  

(4.9)

The Hamilton equations

\[ \ddot{x}_1 + x_1 x_2^2 = 0, \quad \ddot{x}_2 + x_1^2 x_2 = 0 \]  

(4.10)

have unstable "conoidal" periodic solutions

\[ x_1 = x_2 = f, \quad x_1 = -x_2 = f; \quad f = \text{cn}(t, 1/\sqrt{2}). \]  

(4.11)

We consider the two-dimensional section by the hyperplane \( x_2 = 0 \) of the energy integral three-dimensional surface on which the solutions of (4.10) are located. The periodic trajectories (4.11) intersect this section at points which are fixed under the Poincaré mapping. Since they are of hyperbolic type, we can inquire about the mutual disposition of their stable and unstable separatrices. This problem was studied numerically in [57]. The result is illustrated in Fig. 9.

Since the Hamiltonian (4.9) is quasi-homogeneous, precisely the same picture of transversal separatrices occurs on all energy surfaces with a positive value of the total energy. As a consequence, we find that Eqs. (4.10) have no supplementary analytic integral. This result was obtained earlier by Ziglin in [17], using the analysis of the branching of the solutions of system (4.10) in the plane of complex time.
4. Following Chirikov, we consider the "standard" simplectic mapping of the cylinder \((x, \text{ mod } 2\pi, y)\) given by the relations

\[
x' = x + y' \pmod{2\pi}, \quad y' = y + \epsilon \sin x.
\] (4.12)

With \(\epsilon = 0\) we have an integrable mapping: the \(y\) coordinate is the integral and all the points on the circle \(y = \text{const}\) rotate under the mapping through the angle \(y\). Thus, the undisturbed mapping (4.12) has no hyperbolic periodic points. For all \(\epsilon > 0\), however, \(x = y = 0\) is a fixed point of hyperbolic type. The eigenvalues (multiplicators) of the linearized mapping are

\[
1 + \frac{\epsilon}{2} \pm \sqrt{\epsilon + \left(\frac{\epsilon}{2}\right)^2} = 1 \pm \sqrt{\epsilon} + 0(\sqrt{\epsilon}).
\] (4.13)

The point \((x, y) = (\pi, 0)\) is also fixed, though it is of elliptic type for all \(\epsilon > 0\). This remark has a simple interpretation from the point of view of perturbation theory: with small values of the parameter \(\epsilon > 0\), the invariant circle of the undisturbed problem \(y = 0\) is destroyed; from the family of fixed points that compose this circle, a pair of non-degenerate fixed points is born: the coordinates of these points depend analytically on \(\epsilon\) and one of them is stable (to a first approximation), while the other is unstable; the multiplicators of the fixed points can be written as convergent series in powers of \(\sqrt{\epsilon}\) (cf. the conclusion of Theorem 6 of Chapter I).
By (4.13), the stable and unstable separatrices \( \Lambda^+ \) and \( \Lambda^- \) of a fixed hyperbolic point intersect at this point at a small angle of order \( \sqrt{\epsilon} \). It turns out that they intersect close to the point \((x, y) = (\pi, 2\sqrt{\epsilon})\), Lazutkin [58] obtained the asymptotic expression for the angle of intersection of separatrices:

\[
\varphi = \frac{\pi A}{\epsilon} e^{-\frac{x^2}{2\epsilon}} [1 + O(\epsilon^{1/8-\delta})]
\]  

(4.14)

where \( A = 1118, 82770595 \ldots \), \( \delta \) is any positive number, and the constant in the estimate \( O(\ldots) \) depends on \( \delta \). The exponential smallness of the angle \( \varphi \) can be deduced from Neishtadt's results, obtained under very general assumptions [59].

It follows in particular from (4.14) that the separatrices \( \Lambda^+, \Lambda^- \) intersect transversally and that, as a consequence, there is a "stochastic" layer close to \( \Lambda^+ \cup \Lambda^- \). Chirikov earlier proved the existence of this layer by numerical computations, and that its size increases as \( \epsilon \) increases [60]. On further increase of \( \epsilon \) the layer merges with other stochastic layers of the same origin. However, Lazutkin's basic result still remains the asymptotic expression (4.14), which is unique in problems of this kind. He obtained (4.14) by continuation of the mapping (4.12) into the complex plane of variation of variables \( x, y \). Admittedly, as Lazutkin himself remarked, the proof of (4.14) in [58] is not entirely strict and complete. It would be useful to give a strict proof, and also to extend Lazutkin's technique to analytic Hamiltonian systems for which there are no hyperbolic periodic solutions when the value of the perturbing parameter is zero. We discussed such systems in Chapter I.

3. TOPOLOGICAL AND GEOMETRIC OBSTACLES TO INTEGRABILITY

3.1. Natural mechanical systems

Let \( M^n \) be a complete analytic Riemann manifold and \( \Sigma^{2n-1} \) a fiber space of unit tangent vectors. The Riemann metric specifies in \( \Sigma \) a dynamic system which is usually called a geodesic flow. From the point of view of mechanics, a geodesic flow describes the inertial motion of a particle in \( M \) with unit velocity. The familiar Mopertuis principle
reduces the motion of a mechanical system under the action of potential forces to a geodesic flow. Let us recall the statement of this principle. Let $M$ be the space of positions of the mechanical system, $T$ the kinetic energy (Riemann metric in $M$), $V$ the potential of the field of force (function in $M$). The motions are the extremals of the action functional

$$
\int_{t_1}^{t_2} (T - V) dt.
$$

Along the motions the total energy $T + V$ of the system is conserved. With a fixed value $\hbar = T + V$, since $T \geq 0$, the motion occurs in the domain $B_\hbar = \{ V \leq \hbar \}$. The Mopertuis principle asserts that the trajectories of the mechanical system with the reserve $\hbar$ of total energy, which are located inside $B_\hbar$ (i.e., in $B \setminus \delta B$), are geodesic lines of the Jacobi metric

$$(\hbar - V)T.$$

If $\hbar > \max_M V$, the Jacobi metric is the Riemann metric in the whole of $M$ (and not merely inside $B_\hbar$).

**Theorem 1** [61]. If $M$ is a compact two-dimensional surface of genus greater than unity, then the geodesic flow in $\Sigma$ has no non-constant analytic integral.

The proof is based on an analysis of the set of unstable periodic trajectories. Since the genus of surface $M$ is greater than unity, its Gaussian curvature is on the average negative. If the curvature is negative everywhere, the flow in $\Sigma$ is an Anosov system. In this case all the periodic trajectories are unstable, they fill densely $\Sigma$ everywhere, and the geodesic flow has not even a non-constant continuous integral. The fact that the phase trajectories are hyperbolic lies at the basis of the proof, offered by Alekseev [62] and by Libre and Simo [63], that the restricted three-body problem is non-integrable. Note that negative curvature on the average is by no means always negative everywhere.

Theorem 1 can be extended in various ways. A multi-dimensional analog was proved by I.A. Taimanov. We shall say that a geodesic flow in $M^n$ is completely integrable analytically if, in the space of tangent fiber of $M^n$ there exist analytic functions $F_1, \ldots, F_{n-1}$, such that

i) $\{F_i, F_j\} = 0$ for all $1 \leq i, j \leq n - 1$, 
ii) almost everywhere in $\Sigma^{2n-1}$ the functions $F_1, \ldots, F_{n-1}$ are independent,

iii) the bounds of $F_1, \ldots, F_{n-1}$ in $\Sigma^{2n-1}$ are the first integrals of the geodesic flow.

Theorem 2 [64]. Let $M$ be a multi-dimensional compact Riemann manifold and let one of the following conditions hold:

1) $\text{dim } M < \text{rank } H_1(M, \mathbb{Z})$,

2) the fundamental group $\pi_1(M)$ does not contain a commutative subgroup of finite index.

Then, the geodesic flow in $\Sigma$ is not completely integrable analytically.

For $n = 2$ the two conditions of Theorem 2 are equivalent. A sufficient condition for non-integrability in the form of inequality (1) was in fact proved in [61] in the case $n = 2$, and was stated as a hypothesis by the present author in [8]. The second condition is new. It would be interesting to find other bounds on the topology of the space of positions of a completely integrable natural mechanical system. Assume, in particular, that the analytic manifold $M$ does not satisfy conditions (1) and (2) of the theorem; we ask whether there is always a completely integrable mechanical system with the space of positions $M$. The answer seems to be in the negative. The proof of Theorem 2 given in [64] is of a purely topological kind; as distinct from the proof given in [61] in the case $n = 2$, it gives no significant information about effects of a dynamic kind that prevent integrability.

Another possible way of extending Theorem 1 is to consider domains with a geodesically convex boundary. Assume that $\tilde{M}$ is a compact submanifold with an edge in the analytic two-dimensional surface $M^2$. Denote by $\tilde{\Sigma}$ the set of all points on $\Sigma^3$, which transform under the projection $\pi: TM^2 \to M^2$ into points of $M$. We shall say that $\tilde{M}$ is geodesically convex if, given any two close points on the boundary $\partial \tilde{M}$, the shortest geodesic joining them lies entirely in $\tilde{M}$.

Theorem 3. Let $\tilde{M}$ be a geodesically convex submanifold with negative Euler characteristic. Then, the geodesic flow in $\Sigma$ has no non-constant analytic integral. Moreover, there is certainly no analytic integral in any neighbourhood of the set $\tilde{\Sigma}$ in $\Sigma$.

If $\partial M = \phi$, we again obtain Theorem 1. Theorem 3 was first proved by the author under the assumption that $\text{rank } H_1(\tilde{M}, \mathbb{Z}) > 2$. Later,
Bolotin replaced this by the weaker condition $\chi(M) < 0$, where $\chi$ is the Euler characteristic [65]. The proof of Theorem 3 is based on the method of [61]. It uses the fact that, in every class of freely homotopic paths in $M$, there is an unstable closed geodesic. The existence of closed geodesics (without considering stability) in manifolds with convex boundary was pointed out in the classical works of Whittaker [66] and Birkhoff [2].

Following Bolotin, we apply these general results to systems with a potential of Newtonian type. Let $\mathcal{M}^2$ be the space of positions of a natural system with two degrees of freedom, and $V: \mathcal{M} \to \mathbb{R}$ the potential energy. We say that $V$ is a potential of Newtonian type if it is an analytic function except for a finite number of points $z_1, \ldots, z_n$, and in conformal coordinates $z$ (with respect to the metric specified by the kinetic energy) with origin at the singular point $z_s$, $V$ has the form

$$-f(z)/|z|$$

where the function $f$ is analytic in the neighbourhood of the point $z_s$ and $f(0) > 0$.

**Theorem 4** [67]. Let $\mathcal{M}$ be compact, and let the potential $V$ have $n > 2\chi(M)$ singular points. Then, with

$$h > \sup_{\mathcal{M}} V,$$

there is no non-constant analytic integral on the energy surface $\Sigma_h = \{H = h\}$.

If $n = 0$, we obtain Theorem 1. The condition is violated only if

(i) $\mathcal{M}$ is a sphere and $n \leq 4$,

(ii) $\mathcal{M}$ is a projective plane and $n \leq 2$,

(iii) $\mathcal{M}$ is a torus or Klein bottle and $n = 0$.

In the non-compact case we need auxiliary conditions on the behaviour of the kinetic energy at infinity. Assume that $\chi(M) \neq -\infty$; then we know from topology that $\mathcal{M}$ can be transformed into a compact surface $\overline{\mathcal{M}}$ by adding a finite number of points at infinity $\overline{z}_s$. Let $D_s \subset \overline{\mathcal{M}}$ be neighbourhoods of the points $z_s$, diffeomorphic to discs. An extra assumption is that any closed curve in $D_s$ which embraces the point $\overline{z}_s$ cannot be contracted to the point $\overline{z}_s$ in the class of curves of bounded length (in the metric specified by the kinetic energy). We also assume that $\sup_{\mathcal{M}} V < \infty$. 
Theorem 5 [67]. Let $M$ be non-compact, let the kinetic energy satisfy the above condition at infinity, and let the potential energy have $n > 2 \chi(M)$ singular points. Then, with $h > \sup_M V$, the system has no analytic integrals on the surface $\Sigma_h = \{H = h\}$.

Theorems 4 and 5 are proved by means of Theorem 3, using Levi-Civita regularization. As an illustration, we take the motion of a point on the plane in the gravitational field of $n$ fixed centres. Let $z_1, \ldots, z_n$ be distinct points of the complex plane $C$. The Hamiltonian of the problem of $n$ centres has the form

$$H(p, z) = \frac{1}{2} |p|^2 + V(z), \quad p \in C, \quad z \in C \setminus \{z_1, \ldots, z_n\},$$

where

$$V(z) = -\sum_{k=1}^{n} \mu_k |z - z_k|^{-1}, \quad \mu_k > 0$$

is the gravitational potential. Here, $M = C$, $\chi(M) = 1$, the kinetic energy (Euclidean metric in $M$) satisfies the necessary condition at infinity, and the potential $V < 0$. Hence, by Theorem 5, with $n > 2$ the problem of $n$ centres is non-integrable on the energy surface $H > 0$. Notice that the classical integrable Kepler and Euler problems correspond to the cases $n = 1$ and $n = 2$.

Let us outline the proof. Let $R$ be the Riemann surface of the function $\sqrt{(z - z_1) \ldots (z - z_n)}$, and $\pi: R \to C$ a projection. It can be shown that Levi-Civita regularization (passage from the space of positions $M = C$ to the Riemann surface $R$) reduces the phase flow on the energy surface $H = h > 0$ to the geodesic flow on $R$ with a total metric. Let $D$ be a disc on the complex plane $C$ with a sufficiently large radius. Then, the set $\hat{R} = \pi^{-1}(D)$ is compact, geodesically convex, and homotopically equivalent to $R$. By the Riemann-Hurwicz formula, $\chi(R) = 2 - n < 0$ if $n > 2$. It remains to use Theorem 3.

It turns out that the conditions of Theorems 4 and 5 cannot be weakened. Let $M$ be a two-dimensional surface, $T$ the kinetic energy on $M$, $z_1, \ldots, z_n$ are points of $M$. We fix the value $h$ of the total energy. We have:

Theorem 6 [13]. There is a potential $V$ of Newtonian type with singularities at the points $z_1, \ldots, z_n$, such that the Hamilton system
with Hamiltonian $H = T + V$ has a supplementary analytic integral, quadratic in the momenta, where

(i) if $M$ is not compact, then $V < h$;

(ii) if $M$ is compact and $n > 2\chi(M)$, then $V < h$ everywhere except for a finite number of points;

(iii) if $M$ is compact and $n \leq 2\chi(M)$, then $\sup_M V < h$;

(iv) if $M$ is not compact, while $T$ is Euclidean at infinity and $n \leq 2\chi(M)$, then $\sup_M V < h$.

The condition that the Riemann metric $T$ be Euclidean at infinity implies the following. Let $M$ be obtained from the compact Riemann surface $\tilde{M}$ by discarding a finite number of points at infinity $\tilde{z}_s$. Then, in the neighbourhood of every point $\tilde{z}_s$ in conformal coordinates on $\tilde{M}$, the metric $T$ is Euclidean.

### 3.2. Systems with gyroscopic forces

It happens that the result of the previous section can be extended with certain modifications and additions to the more complex case of systems with gyroscopic forces. These systems can be of very diverse kinds. Gyroscopic forces arise on transition to a rotating reference system, on reducing the number of degrees of freedom of a system with symmetry (cf. e.g., [8], Chapter III), or when describing the motion of charged particles in a magnetic field. Let us give the formal definition.

Let $M$ be the space of positions of a natural system, $x_1, \ldots, x_n$ be local coordinates in $M$, and $y_1, \ldots, y_n$ be momenta. The coordinates $x, y$ are canonical and in them the simplectic structure $\omega$ has the standard form

$$\omega = \sum dy_i \wedge dx_i.$$  

We additionally consider the closed 2-form in $M$:

$$f = \sum f_{ij}(x) \, dx_i \wedge dx_j, \quad df = 0.$$  

In mechanics this form is called the form of the gyroscopic forces. The sum of the two forms $\omega + f$ defines a new simplectic structure in the space of co-tangent bundle of the manifold $M$. If $H$ is a function in $T^*M$, then the pair $(\omega + f, H)$ gives us a Hamilton system with Hamiltonian $H$; we call it a system with gyroscopic forces. Clearly, the
presence of gyroscopic forces does not alter the total energy $H$. We can apply the Darboux theorem to the form $\omega + f$ and write it in the canonical form. For this, using the fact that the form $f$ is closed, we write locally $f = dF, F = \Sigma F_k(x)dx_k$. Then, in variables $x, y$, we have

$$\omega + f = \Sigma dy_i \wedge dx_i + \Sigma df_i \wedge dx_i = \Sigma d(y_i + F_i) \wedge dx_i.$$  

Consequently, the variables $x', y'$ given by the equations

$$x'_k = x_k, \quad y'_k = y_k + F_k(x_1, \ldots, x_n); \quad 1 \leq k \leq n,$$

will be canonical coordinates for the new simplectic structure. In the new variables the Hamilton equations will have the canonical form with Hamiltonian $H(x', y' - F) = H(x, y)$.

Take the case when the closed form of the gyroscopic forces is exact: $f = dg$, where $g$ is a 1-form on $M$. In this case the equations of motion can be written as Lagrange equations with the globally defined Lagrangian

$$L = \frac{1}{2} \langle \dot{x}, \dot{x} \rangle + \langle v(x), \dot{x} \rangle - V(x).$$

Here, the metric $\langle , \rangle$ gives the kinetic energy of the system, the 1-form $\langle v, \dot{x} \rangle$ is precisely the same as the form $g$, and $v$ is a vector field in $M$. The Hamilton function is $H = H_2 + H_1 + H_0$, where $H_s$ is a homogeneous form in the momenta of degree $s$, while

$$H_0 = \frac{1}{2} \langle v, v \rangle + V. \quad (2.1)$$

Now let $M$ be a two-dimensional oriented manifold and $\varphi$ the 2-form of area in $M$. Clearly, any form of gyroscopic forces has the form $\lambda \varphi$, where $\lambda$ is a function in $M$. We shall say that the form $f = \lambda \varphi$ preserves its sign if $\lambda \geq 0 (\leq 0)$ everywhere in $M$. This last is true if $f \equiv 0$ (i.e., the system is reversible). The equations of motion of systems with two degrees of freedom were reduced by Birkhoff to the form [2]

$$x'' + \lambda y' = -\frac{\partial V}{\partial x}, \quad y'' - \lambda x' = -\frac{\partial V}{\partial y}. \quad (2.2)$$

Here, $x$ and $y$ are local isothermal coordinates in $M$, and the prime denotes differentiation with respect to new time. More precisely, Eqs. (2.2) describe the trajectories of motion of the system with a fixed
(say zero) value of the total energy. We must therefore add to Eqs. (2.2) the energy integral

\[ \frac{x'^2 + y'^2}{2} + V = 0. \]

It is well known that, for the case of the two-dimensional torus, the isothermal coordinates can be introduced in the large.

Following Birkhoff, we take the problem of the existence of conditional polynomial integrals. By definition, the first integral of the system, given on a fixed level surface of the energy integral, is called a conditional polynomial integral if it can be continued up to a function in \( T^*M \) which is a polynomial in the momenta with coefficients that are one-valued in \( M \).

We again assume that the potential \( V \) of Newtonian type has \( n \) singular points in \( M \). We shall assume that all objects are analytic.

**Theorem 7** [13]. Let \( M \) be compact and \( n > 2\chi(M) \). Then, with \( h > \sup_M V \), there is no non-constant first integral, polynomial in the momenta, at the energy level \( H = h \).

If \( n = 0 \) and \( f \equiv 0 \), this is V.N. Kolokol'tsev's theorem [68]. As a matter of fact, if, for any domain \( D \subset M \) with a non-empty edge, we have the inequality

\[ \iint_D f + \int_{\partial D} 2\sqrt{(h - V)T} \, dt > 0 \quad (2.3) \]

then the polynomial integrals in Theorem 7 can be replaced by analytic [67]. If the form of the gyroscopic forces is exact, then condition (2.3) certainly holds when \( h > \sup_M H_0 \), where the function \( H_0 \) is given by (2.1). With \( f \equiv 0 \), we obtain Theorem 4 of Section 1.

**Theorem 8** [13]. Let \( M \) be non-compact, let the kinetic energy be Euclidean at infinity, and let the potential \( V \) have \( n > 2\chi(M) \) singular points \( z_1, \ldots, z_n \). Then, there are no first integrals in \( T^*(M \setminus \{z_1, \ldots, z_n\}) \) which are polynomial in the momenta and independent of the function \( H \).

In fact, if the form \( f \) is exact and we have \( h > \sup_M H_0 \), then, with \( n > 2\chi(M) \), there is not even an analytic non-constant integral on the surface \( H = h \) [67]. With \( f \equiv 0 \), we again obtain Theorem 5.
As an example, take the plane circular many-body problem. Let the \( n \) points \( z_1, \ldots, z_n \) be clamped in the plane \( M \), and rotate about a fixed point 0 with constant angular velocity \( \Omega \) (the vector \( \Omega \) is orthogonal to \( M \)), and let the point \( z \) of unit mass move in \( M \) under the action of forces of gravitational attraction towards the points \( z_1 \ldots, z_n \). Here,

\[
L = \frac{1}{2} |\dot{z}|^2 + (\Omega \times z, \dot{z}) - V,
\]

\[
V = - \sum_{k=1}^{n} \frac{\mu_k}{|z - z_k|} - \frac{\Omega^2 |z|^2}{2}, \quad \mu_k > 0.
\]

The restricted many-body problem is integrable with \( n = 1 \) and all \( \Omega \) (Kepler's problem), and also with \( n = 2 \) and \( \Omega = 0 \) (Euler's problem of two fixed centres). With \( n > 2 \) and all \( \Omega \), the problem has no supplementary analytic integrals. For,

\[
H_0 = V + \frac{|\Omega \times z|^2}{2} = - \sum \frac{\mu_k}{|z - z_k|} < 0
\]

and \( f = 2|\Omega|\varphi \), where \( \varphi \) is the standard form of the area on the plane \( M \). Since \( \chi(M) = 1 \), then, with \( n > 2 \) and \( h > 0 \), the restricted \( n \)-body problem has no analytic integral on the energy surface \( H = T + V = h \).

For \( n = 2 \) and \( \Omega \neq 0 \) (restricted three-body problem), such a claim has not been proved. What is more, there is Shatz's familiar hypothesis about the integrability of the three-body problem for positive values of the total energy (see [69]). This hypothesis is linked with a more general idea: in the particle scattering problem with a non-compact space of positions, the data at infinity (e.g., the particle momenta) are candidates for the role of first integrals. However, the realization of this idea runs into difficulties of a fundamental kind, connected with the domain of definition and the smoothness of the "scatter integrals." One such difficulty is the possibility of capturing in the problem many interacting particles.

In the restricted three-body problem, weaker results concerning non-integrability are known. Poincaré proved that there are no auxiliary integrals, analytic in the "heavy" particle masses \( \mu_1 \) and \( \mu_2 \) [14]. Libre and Simo [63] used Alekseev's method of quasi-random motions to show that there is no new analytic integral provided that the mass of one
body (say $\mu_i$) is small and the negative constant energy integral is sufficiently large in absolute value. Apart from this, there is Ziglin’s celebrated result on the absence of new algebraic first integrals [70]. This was proved by Bruns’ method. It appears that the restricted three-body problem admits no first integrals which are polynomial in the momenta and are independent of the energy integral.

With $n = 2\chi(M)$, the structure of the gyroscopic forces is the definitive factor in the integrability of the Hamiltonian system.

**Theorem 9 [13].** Let $M$ be compact, and let the Newtonian potential have $2\chi(M)$ singular points. If

$$\int_M f \neq 0,$$

then there is no conditional polynomial integral at the level $H = h$, where $h > \sup_M V$.

As is shown by the example of the problem on the motion of a charge over a plane torus in a constant magnetic field (see the Introduction), under the conditions of Theorem 9 there can exist analytic first integrals.

In the non-compact case it is not possible to write existence conditions for a supplementary polynomial integral in the form of topological constraints.

To illustrate Theorem 9, we take the equations in the n-dimensional torus $T^n = \{ x \mod 2\pi \}$ with supplementary forces of gyroscopic type:

$$\ddot{x} = \Lambda(x) \dot{x} - \frac{\partial V}{\partial x}. \quad (2.4)$$

Here, $\Lambda$ is a skew-symmetric matrix, $2\pi$-periodic with respect to $x$, and $V$ is the potential of the field of force. Birkhoff’s equations (2.2) obviously have the form (2.4). We consider the existence of polynomial integrals in the velocities with one-valued coefficients which are independent of the energy integral

$$H = \frac{1}{2} (\dot{x}, \dot{x}) + V(x).$$

We use Poincaré’s method. For this, we introduce new time $t \mapsto t/\epsilon$, where $\epsilon$ is a parameter. After this substitution, Eqs. (2.4) become
\[ \dot{x} = \epsilon \Lambda \dot{x} - \epsilon^2 \frac{\partial V}{\partial x} \] (2.5)

which contain the small parameter \( \epsilon \). Since the velocity \( \dot{x} \) transforms to \( \dot{x}/\epsilon \), the polynomial integral of Eqs. (2.4) transforms to the integral of Eqs. (2.5), analytic in:

\[ F_0(\dot{x}, x) + \epsilon F_1(\dot{x}, x) + \ldots \]

The function \( F_0 \) is obviously \( 2\pi \)-periodic in \( x \). The function \( F_0 \) is the first integral of the unperturbed system \( \ddot{x} = 0 \). Consequently,

\[ \dot{F}_0 = \left( \frac{\partial F_0}{\partial \dot{x}}, \dot{x} \right) + \left( \frac{\partial F_0}{\partial x}, \ddot{x} \right) = \left( \frac{\partial F_0}{\partial x}, \dot{x} \right) = 0. \]

Hence we conclude that \( F_0 \) is independent of the angular variables \( x \). The function \( F_1 \) satisfies the equation

\[ \left( \frac{\partial F_0}{\partial \dot{x}}, \Lambda \dot{x} \right) + \left( \frac{\partial F_1}{\partial x}, \dot{x} \right) = 0. \]

On averaging this equation over \( x_1, \ldots, x_n \), we obtain

\[ \left( \frac{\partial F_0}{\partial \dot{x}}, \Lambda_0 \dot{x} \right) = 0, \quad \Lambda_0 = \frac{1}{(2\pi)^n} \int_{\Gamma^n} \Lambda d^n x. \]

Thus the function \( u \mapsto F_0(u) \) is the first integral of a linear system with constant coefficients \( \dot{u} = \Lambda_0 u \). If we assume that system (2.4) has \( n \) independent polynomial integrals, then \( \Lambda_0 = 0 \). This conclusion confirms the conclusion of Theorem 9. Let \( \pm i\lambda_1, \ldots, \pm i\lambda_m \) be the eigenvalues of the skew-symmetric matrix \( \Lambda_0 \) (some of them may be zero). Assume that, among the real \( \lambda_1, \ldots, \lambda_m \) there are \( k \) numbers that are independent over the field \( Q \). Then the system (2.4) can have at most \( n - k \) independent one-valued integrals.

3.3. Linear and quadratic integrals

It is well known that the existence of first integrals, linear in the momenta, is closely linked with the symmetry groups that act in the space of positions, and the existence of integrals, quadratic in the momenta, with separation of the canonical variables. In this section we discuss the existence conditions for linear and quadratic integrals on the whole.
1. It turns out that the existence of linear integrals imposes restrictions, not only on the Riemann metric (the kinetic energy) and the potential of the field of force, but also on the topology of the space of positions.

*Theorem 10* [71]. Let $M$ be a connected, compact, oriented even-dimensional manifold. If the natural Hamiltonian system in $T^*M$ has $k \geq (\dim M)/2$ independent linear integrals, pairwise in involution, then $\chi(M) \geq 0$.

In particular, if $\dim M = 2$, there can only be a linear integral when $M$ is diffeomorphic to a sphere or torus. The proof of Theorem 10 is based on using Kobayashi's results on the action of commutative groups of isometries on a Riemann manifold [72].

2. We now take an irreversible system with a two-dimensional torus as configuration space. By refining Birkhoff's classical result ([2], Chapter II), we can indicate a criterion for the existence of a "many-valued" linear integral. By a many-valued integral we mean a closed 1-form in the phase space, whose derivative along the vector field vanishes. It is desirable to consider many-valued integrals for two reasons:

(1) in the simplest irreversible systems there can be integrals, polynomial in the velocities, with many-valued coefficients,

(2) Liouville's theorem on completely integrable systems can easily be extended to the case when instead of ordinary integrals, closed 1-forms are considered.

*Proposition 1*. Assume that the system has a conditional linear integral (possibly many-valued) on the energy surface $H = h$, where $h > \max V$. Then, in the space of positions we can choose angular coordinates $x_1, x_2 \mod 2\pi$ and make a change of time $dt = \xi(x_1, x_2) \, d\tau$ in such a way that the trajectories with margin of total energy $h$ describe a Hamiltonian system for which

(i) the kinetic energy is a quadratic form in $x_1', x_2'$ with constant coefficients,

(ii) the form of the gyroscopic forces $f$ has the form $\lambda(x_1)dx_1 \wedge dx_2$,

(iii) the potential is independent of the variable $x_2$.

In the new variables $x_1, x_2, \tau$ the Lagrangian has the form
\[ L = \frac{1}{2} \sum a_{ij} x_i' x_j' + \mu(x_1) x_1' - V(x_1), \quad \mu = \int \lambda(x) \, dx. \]

This function is one-valued only if
\[ \int_{T^2} f = 0. \]

The variable \( x_2 \) is cyclical: it does not appear in the expression for the Lagrange function. Corresponding to it we have the cyclical integral
\[ \frac{\partial L}{\partial x_2'} = \sum a_{2i} x_i' + \mu(x_1) \]
linear in the velocities. Proposition 1 does not claim that this integral is the same as the linear integral which is specified right from the start. To quote a simple counter-example: the reversible system with kinetic energy \( T = (\dot{x}_1^2 + \dot{x}_2^2)/2 \) and zero potential has the linear integral \( \dot{x}_1 + \sqrt{2} \dot{x}_2 \), which cannot be made cyclical no matter how the angular coordinates are chosen.

To prove the proposition we use Birkhoff's equations (2.2). Let
\[ \ell x' + my' + n - \]
be a conditional linear integral. The differentials \( d\ell, dm, \) and \( dn \) are one-valued on \( T^2 = \{ x, y \ mod \ 2\pi \} \). We evaluate the derivative of integral (3.1) in the light of system (2.2):
\[ \frac{\partial \ell}{\partial x} x'^2 + \left( \frac{\partial \ell}{\partial y} + \frac{\partial m}{\partial x} \right) x'y' + \frac{\partial m}{\partial y} y'^2 + \]
\[ + \left( m\lambda + \frac{\partial n}{\partial x} \right) x' + \left( -\ell \lambda + \frac{\partial n}{\partial y} \right) y' - \ell \frac{\partial V}{\partial x} - m \frac{\partial V}{\partial y}. \]
(3.2)

Since the function (3.1) is the integral of Eqs. (2.2) on the energy surface
\[ x'^2 + y'^2 + 2V = 0, \]
the leading form of polynomial (3.2) must be divisible by the Hamilton function. Hence we find that
\[ \frac{\partial \ell}{\partial x} - \frac{\partial m}{\partial y} = 0, \quad \frac{\partial \ell}{\partial y} + \frac{\partial m}{\partial x} = 0. \]
Consequently, the forms \( dl \) and \( dm \) are harmonic in \( T^2 \) and hence

\[
m = ax + by + m_0, \quad \ell = bx - ay + \ell_0;
\]

\( a, b, m_0, \ell_0 = \text{const.} \)

On equating to zero the coefficients of \( x' \) and \( y' \) in (3.2), we obtain

\[
\frac{\partial n}{\partial x} = -m\lambda, \quad \frac{\partial n}{\partial y} = \ell\lambda. \tag{3.3}
\]

Since the form \( dn \) is one-valued, we find that, if \( \lambda \neq 0 \), then \( a = b = 0 \), so that \( m = m_0, \ell = \ell_0 \). This is the case considered below. On equating (3.2) to zero, we obtain the further relation on the potential

\[
\ell_0 \frac{\partial V}{\partial x} + m_0 \frac{\partial V}{\partial y} = 0. \tag{3.4}
\]

From (3.3) we obtain a similar relation for the function \( \lambda \):

\[
\ell_0 \frac{\partial \lambda}{\partial x} + m_0 \frac{\partial \lambda}{\partial y} = 0. \tag{3.5}
\]

If the numbers \( \ell_0 \) and \( m_0 \) are rationally incommensurable, it follows from (3.4) and (3.5) that the functions \( V \) and \( \lambda \) are constant, and the proposition is proved. Let \( m_0/\ell_0 = p/q \), where the integers \( p \) and \( q \) are relatively prime. It is easily seen from (3.4) and (3.5) that

\[
V = \hat{V}(px - qy), \quad \lambda = \hat{\lambda}(px - qy),
\]

where the functions \( \hat{V} \) and \( \hat{\lambda} \) are \( 2\pi \)-periodic. We perform the linear transformation

\[
x_1 = px - qy, \quad x_2 = ux + vy
\]

with integers \( u, v \) that satisfy the relation \( pu + qv = 1 \). Since \( p \) and \( q \) are relatively prime, such numbers \( u, v \) exist. We see that \( x_1, x_2 \mod 2\pi \) are the required variables.

3. We now take a reversible system with two degrees of freedom, whose space of positions is again the two-dimensional torus.

**Proposition 2.** Assume that the system has a conditional quadratic integral with one-valued coefficients on the surface \( H = h \), where \( h > \max V \). Then, in the space of positions, we can choose angular coordinates \( x_1, x_2 \mod 2\pi \) and make the change of time
\[ \frac{\text{d}t}{\xi(x_1, x_2)} \text{d}\tau \] in such a way that the trajectories of motion with energy \( h \) are described by a Lagrangian system with Lagrangian

\[ \frac{1}{2} (x_1'^2 + x_2'^2) + \eta(px_1 + qx_2) + \zeta(qx_1 - px_2) \]  

(3.6)

where \( \eta(\cdot) \) and \( \zeta(\cdot) \) are \( 2\pi \)-periodic functions and \( p, q \) are integers.

We make the linear change of variables

\[ x = px_1 + qx_2, \quad y = qx_1 - px_2. \]

In the new variables the Lagrangian (3.6) becomes

\[ \frac{x}{2} (x'^2 + y'^2) + \eta(x) + \zeta(y), \quad x^{-1} = p^2 + q^2. \]

The variables \( x \) and \( y \) are separable: the functions

\[ \frac{x}{2} x'^2 + \eta(x), \quad \frac{x}{2} y'^2 + \zeta(y) \]

are independent quadratic integrals.

Proposition 2 is a global version of Birkhoff's familiar result concerning conditionally quadratic integrals ([2], Chapter II).

For the proof, we again use Eqs. (2.2), in which we have to put \( \lambda \equiv 0 \). We assume that Eqs. (2.2) have the quadratic integral

\[ \frac{1}{2} (ax'^2 + 2bx'y' + cy'^2) + dx' + ey' + f \]  

(3.7)

on the surface

\[ x'^2 + y'^2 = 2V. \]  

(3.8)

Let us write the terms of third degree in the velocities in the expression for any integral (3.7) in the new time \( \tau \):

\[ \frac{1}{2} \frac{\partial a}{\partial x} x'^3 + \left( \frac{\partial b}{\partial x} + \frac{1}{2} \frac{\partial a}{\partial y} \right) x'^2y' + \left( \frac{\partial b}{\partial y} + \frac{1}{2} \frac{\partial c}{\partial x} \right) x'y'^2 + \]

\[ + \frac{1}{2} \frac{\partial c}{\partial y} y'^3. \]

Since (3.7) is a conditional integral, this polynomial must be divisible by \( x'^2 + y'^2 \). Consequently,
\[ \frac{\partial}{\partial x} (a - c) - \frac{\partial (2b)}{\partial y} = 0, \quad \frac{\partial}{\partial y} (a - c) + \frac{\partial (2b)}{\partial x} = 0, \]

and hence the functions \( a - c \) and \( b \) are harmonic in the two-dimensional torus. By our assumption that they are one-valued, \( a - c = \text{const}, \ b = \text{const} \).

On using this fact and the energy integral (3.8), we can transform the quadratic integral (3.7) to a form in which the coefficients \( a, b, c \) are constant. On again differentiating the integral (3.7) in the light of the system and equating to zero the coefficients of \( x' \) and \( y' \), we obtain the relations

\[ a \frac{\partial V}{\partial x} + b \frac{\partial V}{\partial y} = \frac{\partial f}{\partial x}, \quad b \frac{\partial V}{\partial x} + c \frac{\partial V}{\partial y} = \frac{\partial f}{\partial y}. \]

Consequently,

\[ (a - c) \frac{\partial^2 V}{\partial x \partial y} + b \left( \frac{\partial^2 V}{\partial y^2} - \frac{\partial^2 V}{\partial x^2} \right) = 0. \quad (3.9) \]

Writing the potential \( V \) as the Fourier series

\[ \Sigma v_{mn} \exp [i(mx + ny)], \]

we obtain from (3.9) the series of equations

\[ [(a - c) mn + b (n^2 - m^2)] v_{mn} = 0. \]

Assume that \( v_{m_1 n_1} \neq 0 \) and \( v_{m_2 n_2} \neq 0 \). Then,

\[ (a - c) m_1 n_1 + b (n_1^2 - m_1^2) = (a - c) m_2 n_2 + b (n_2^2 - m_2^2) = 0. \quad (3.10) \]

Clearly, neither of the numbers \( a - c \) or \( b \) is zero. Otherwise, the integral (3.7) reduces to a linear integral. From (3.10) we find that

\[ \frac{m_1 n_1}{n_1^2 - m_1^2} = \frac{m_2 n_2}{n_2^2 - m_2^2}. \]

Hence we conclude that either \( m_1/n_1 = m_2/n_2 \), or \( (m_1/n_1) \cdot (n_2/m_2) = -1 \). Thus the integervalued vectors \( (m_1, n_1) \) and \( (m_2, n_2) \) are either parallel or orthogonal. The proposition is proved. Geodesic flows as a whole in the two-dimensional sphere and torus with a quadratic integral have been studied by V.N. Kolokol'tsev [68]. Little
is known about the existence conditions of polynomial integrals of degree $\geqslant 3$.

3.4. **Morse theory of integrable Hamiltonian systems**

Another approach to the study of topological obstacles to complete integrability of Hamiltonian systems was proposed by A.T. Fomenko [73]. He linked the existence of a supplementary smooth integral of general position with the topology of the level surface of the energy integral and the number of stable closed trajectories.

We turn to the exact statement. Let $M^4$ be the phase space of the Hamiltonian system with two degrees of freedom, and let $H$ be the Hamilton function. We consider the fixed compact three-dimensional surface $\Sigma^3 \subset M$ of the non-singular level of Hamiltonian $H$. We assume that the Hamiltonian system has a smooth integral $f$ on $\Sigma$. We call $f$ a Morse integral if its critical points form on $\Sigma$ a non-degenerate critical submanifold $N$, i.e., the Hessian $d^2f$ is non-degenerate in subspaces which are transversal to these submanifolds. Analysis of concrete integrated problems of classical mechanics shows that, in all the known cases, the supplementary integrals prove to be Morse. Non-critical level surfaces of the function $f$ are always oriented (we use the fact that the symplectic manifold of $M$ is oriented, and the surface $\Sigma$ is non-critical). The critical surfaces $N$ may not be oriented. In this connection it seems natural to call the integral $f$ oriented if all the critical manifolds are oriented. It can be shown that, by passing to a suitable two-sheeted covering over $\Sigma$, the integral $f$ can always be made oriented.

We call a closed trajectory of the Hamiltonian system on $\Sigma$ stable, if a tubular neighbourhood of it in $\Sigma$ stratifies entirely into two-dimensional concentric invariant tori (non-critical level surfaces of function $f$). Clearly, if $f$ has a strict local maximum or minimum at points of a closed trajectory, this trajectory is stable. The example of a geodesic flow on a plane two-dimensional torus shows that not every completely integrable system has stable trajectories.

**Theorem 11** [73]. Assume that the Hamiltonian system has an oriented Morse integral $f$ on the surface $\Sigma$. Then, if the homology group $H_1(\Sigma, \mathbb{Z})$ is finite or the rank of the fundamental group $\pi_1(\Sigma)$ is equal to one, then the Hamiltonian system has on $\Sigma$ at least two stable closed
trajectories and \( f \) has a strict local minimum or maximum on each of these trajectories.

In the case of a geodesic flow on a plane torus \( T^2 \) we have \( \Sigma = T^3 \) and hence rank \( \pi_1(\Sigma) = 3 \). Let us quote an example of a Hamiltonian system for which there are just two stable closed trajectories on the energy surfaces \( \Sigma \). We take the biharmonic oscillator, whose dynamic behaviour is described by the equations

\[
\ddot{x}_1 + \omega_1^2 x_1 = 0, \quad \ddot{x}_2 + \omega_2^2 x_2 = 0; \quad \omega_1/\omega_2 \not\in \mathbb{Q}.
\]

The energy surface

\[
\dot{x}_1^2 + \dot{x}_2^2 + \omega_1^2 x_1^2 + \omega_2^2 x_2^2 = 2h
\]

with \( h > 0 \) is diffeomorphic to the three-dimensional sphere; hence \( H_1(\Sigma, Z) = 0 \). For all \( h > 0 \) there are just two stable periodic solutions

\[
x_1 = \sqrt{2h} \sin \omega_1 t, \quad x_2 = 0
\]

\[
x_1 = 0, \quad x_2 = \sqrt{2h} \sin \omega_2 t.
\]

**Corollary.** Assume that the group \( H_1(\Sigma, Z) \) is finite and that there are no stable closed trajectories on the energy surface \( \Sigma \) of the Hamiltonian system. Then, the system has no supplementary Morse integral on the surface \( \Sigma \). An application of this claim may be mentioned. It follows from the results of Anosov, Klingenberg, and Takens that, in the set of all geodesic flows on smooth Riemann manifolds, there is an open everywhere dense subset of flows without stable periodic trajectories, see [74]. Thus the property whereby a geodesic flow does not have stable periodic trajectories is a property of general position. We consider geodesic flows on the two-dimensional sphere. In this case, \( M = T^*S^2 \), \( \Sigma = \text{SO}(3) \), and \( H_1(\Sigma, Z) = \mathbb{Z}_2 \). Hence the geodesic flow of general position on the two-dimensional sphere has no supplementary Morse integral on non-singular energy surfaces.

Fomenko has studied in detail the structure of the three-dimensional manifolds of energy integral levels in integrable systems and has found the topological invariants of integrable systems whereby the non-isomorphic systems can be effectively separated.
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