Fermi-like acceleration and power-law energy growth in nonholonomic systems

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Abstract
This paper is concerned with a nonholonomic system with parametric excitation—the Chaplygin sleigh with time-varying mass distribution. A detailed analysis is made of the problem of the existence of regimes with unbounded growth of energy (an analogue of Fermi’s acceleration) in the case where excitation is achieved by means of a rotor with variable angular momentum. The existence of trajectories for which the translational velocity of the sleigh increases indefinitely and has the asymptotics $\tau^\dagger$ is proved. In addition, it is shown that, when viscous friction with a nondegenerate Rayleigh function is added, unbounded speed-up disappears and the trajectories of the reduced system asymptotically tend to a limit cycle.

Keywords: nonholonomic mechanics, Fermi’s acceleration, Chaplygin sleigh, unbounded speed-up, limit cycle, rotor, viscous friction
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(Some figures may appear in colour only in the online journal)
Introduction

1. The Chaplygin sleigh on a plane is one of the best-known model systems in nonholonomic mechanics. According to Chaplygin [18], the sleigh can be designed to have a knife edge and two absolutely smooth legs attached to a rigid body. In this case, the nonholonomic constraint is achieved by means of the knife edge: the translational velocity at the point of contact of the knife edge is orthogonal to its plane (that is, to the body-fixed direction). A similar constraint can also be obtained by using a wheel pair [9] instead of a knife edge. The free dynamics of the Chaplygin sleigh on a horizontal plane was studied by Carathéodory [17]. Depending on the position of the center of mass relative to the knife edge, the sleigh moves in a circle or asymptotically tends to rectilinear motion. In the latter case, we have the classical scattering problem, for which the scattering angle was found in [41]. It is calculated explicitly, since the free motion of the sleigh is integrable and regular [17]. The dynamics of the Chaplygin sleigh on an inclined plane is no longer integrable and exhibits random asymptotic behavior depending on initial conditions [13]. The recent paper [29] investigates the motion of the Chaplygin sleigh under the action of random forces which model a fluctuating continuous medium. It turns out that in this case the sleigh exhibits complex intricate behavior, which, according to the authors, resembles random walks of bacterial cells with some diffusion component. Similar behavior is exhibited by the sleigh under the action of periodic pulsed torque impacts, which depend on the orientation of the sleigh, and in the presence of viscous friction [12]. In [33, 34], the motion of the Chaplygin sleigh with servoconstraints is explored. The paper [5] is concerned with the dynamics of the Chaplygin sleigh with a freely moving material point. Other generalizations of the Chaplygin sleigh problem are discussed in [2, 6, 15].

2. In this paper, we consider various aspects of the dynamics of a nonautonomous Chaplygin sleigh (i.e. with time-varying mass distribution). A detailed analysis is made of the sleigh with a gyrostatic momentum periodically changing with time. In practice this can be achieved by means of a rotor placed inside the body. Our investigation of the dynamics of the Chaplygin sleigh with periodically time-dependent parameters is closely related to the control problem. Since the sleigh can be designed to be a two-wheeled robot [9], it is of great practical importance, since the regimes arising at fixed values of the angular velocities of eccentrics can be taken as basic regimes (called gaits), which the body reaches after various maneuvers initiated by the control system. We note that periodic changes in control functions were also considered in optimal control problems [36, 40]. Problems of controlling the Chaplygin sleigh by displacing the center of mass are addressed in [43] with special emphasis on the maneuver involving a transition from motion in a circle to straight-line motion.

3. In this paper, we examine in detail the dynamics of a reduced system which decouples from a complete system of equations and governs the evolution of the translational and angular velocities of the sleigh. From known solutions of a reduced system the dynamics of the point of contact is defined by quadratures. A reduced system is a system of two (nonlinear) first-order equations with periodic coefficients which govern the evolution of the translational and angular velocities of the sleigh. However, in contrast to Hamiltonian systems with one and a half degrees of freedom, the reduced system possesses no smooth invariant measure [13] and can have different attractors (including strange ones) typical of dissipative systems. In this sense, it is similar to various Duffing and Van der Pol type oscillators with parametric periodic excitation [46] and to the nonlinear Mathieu equation [27, 28]. However, as noted in many publications,
‘nonholonomic dissipation’, which arises due to the divergence being sign-alternating, possesses specific features that require additional research. Starting with [14], strange attractors of different nature [7, 8, 23] are detected in nonholonomic systems. A strange attractor for the Chaplygin sleigh with a material point, which executes periodic oscillations in the direction transverse to the plane of the knife edge, is found in [1, 3].

4. The most interesting problem in the dynamics of the nonautonomous Chaplygin sleigh is that of its speed-up (acceleration). From a physical point of view, interest in it stems from the fact that unbounded growth of energy and hence unbounded speed-up is achieved by means of a mechanism executing small, but regular oscillations.

As noted above, the system considered in this paper differs from Hamiltonian systems with one and a half degrees of freedom. This difference is particularly pronounced in the situation with speed-up.

The Hamiltonian speed-up model began to be discussed in the physical literature in connection with the prediction of Fermi’s acceleration [22] in the Ulam model [47]. As shown numerically in [47] and then proved analytically in [16, 39, 50], acceleration in different variations of the Ulam model is prevented by an invariant curve existing at large velocities and predicted by KAM theory (see also [37]). In the problem of a gravitational machine there exist trajectories for which the particle gathers speed without bound. For a particular case of motion of the lower wall, the book [49] proposes the model of some random process for description of the dynamics. Analysis of this process shows that the velocity increases as a function of time $t^2$. In the general case, the presence of accelerating trajectories is proved by Pustylnikov in [44], where it is also shown that the velocity of the particle at instants of collisions increases as a function of time $t^2$. Causes of the absence of an invariant curve at large velocities in a gravitational machine are discussed in [38].

Thus, in Hamiltonian systems with one and a half degrees of freedom the problem of acceleration reduces to investigating the conditions under which the KAM curves existing in the general case at large energies are destroyed. Among modifications of the Ulam model in which acceleration is observed, we mention generalizations associated with random [26, 30] and piecewise smooth [19] motion of the wall and a relativistic generalization [45].

5. As noted above, nonholonomic systems possess no invariant measure in the general case. Consequently, in the general case, KAM theory cannot be applied to them. This is particularly clearly seen in the problem of the Chaplygin sleigh with periodically changing gyrostatic momentum. It turns out that all solutions of the reduced system are accelerating and have identical asymptotics of the growth of the translational velocity as a function of time $t^3$.

In the gravitational machine it is assumed that the particle collides absolutely elastically with the wall. If one introduces dissipation into this system, assuming the impact to be not absolutely elastic, then acceleration disappears in the gravitational machine [24]. A similar situation is observed in the system considered in this paper. If one introduces the force of viscous friction in the Chaplygin sleigh with variable gyrostatic momentum, then unbounded speed-up disappears also. In this paper we show that all trajectories of the reduced system tend to a limit cycle.

The problem dealt with in this paper shows that in nonholonomic mechanics acceleration is characteristic even of small dimensions. We mention the recent paper [1], in which the speed-up of the Chaplygin sleigh is studied using the averaging method and the asymptotics of the degree of speed-up as a function of time is obtained. We note that the absence
of an invariant measure turns out to be essential and necessary for the presence of speed-up. These issues are very important for developing the control of various mechanical devices.

In [4, 35], the roller-racer (a system consisting of two coupled platforms with one wheeled pair fastened on each of them) is considered. It is shown that, if the angle between the platforms is changed in a prescribed periodic manner, then an unbounded speed-up of the roller-racer is observed for all initial conditions.

The problem we consider here is a model problem, but its analysis allows one to pose the problem of the possibility of speed-up in nonholonomic robots with a more complex control mechanism. In particular, the control of spherical robots is discussed in [10, 11, 31].

The investigation of speed-up in such systems is a complicated and interesting problem.

1. Equations of motion

The Chaplygin sleigh is a platform (rigid body) moving on a horizontal plane with a nonholonomic constraint: at some point the velocity is always orthogonal to a fixed direction (see figure 1). This constraint can be obtained by means of a knife edge rigidly attached to the platform [18] or by means of a wheel pair [9].

To describe the motion of the sleigh, we define two coordinate systems:

- a fixed (inertial) coordinate system $Oxy$;
- a moving coordinate system $Rx_1y_1$ attached to the platform.

We specify the position of the sleigh by the coordinates $(x, y)$ of point $R$ in the fixed coordinate system $Oxy$, and the orientation by the rotation angle $\varphi$. Thus, the configuration space of the system $N = \{ q = (x, y, \varphi) \}$ coincides with the motion group of the plane $SE(2)$.

Let $\mathbf{v} = (v_1, v_2)$ denote the projections of the velocity of point $R$ relative to the fixed coordinate system $Oxy$ onto the moving axes $Rx_1y_1$ and let $\omega$ be the angular velocity of the body. Then

$$\dot{x} = v_1 \cos \varphi - v_2 \sin \varphi, \quad \dot{y} = v_1 \sin \varphi + v_2 \cos \varphi, \quad \dot{\varphi} = \omega.$$  (1)

The constraint equation in this case reads

$$v_2 = 0.$$  (2)

Suppose that $n$ material points $P^i, i = 1, \ldots, n$ move on the platform in a prescribed manner. In this case the kinetic energy of the entire system can be represented in the following form [3]:

$$T = \frac{1}{2} m |\mathbf{v}|^2 + m \omega (c_1(t) v_2 - c_2(t) v_1) + \frac{1}{2} (J(t) + mc_2^2(t)) \omega^2 + m(v_1 \dot{c}_1(t) + v_2 \dot{c}_2(t)) + k(t) \omega,$$

where $m$ is the mass of the entire system, $I(t)$ is its moment of inertia, $e = (c_1(t), c_2(t))$ is the position of the center of mass, and $k(t)$ is the gyrostatic momentum arising from the motion of the points.

For this system the Lagrange equations with undetermined multipliers have the form

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\omega}} \right) = \dot{\mathbf{v}} \cdot \frac{\partial T}{\partial \mathbf{v}}, \quad \frac{d}{dt} \left( \frac{\partial T}{\partial v_1} \right) = \omega \frac{\partial T}{\partial \dot{v}_1} + N,$$

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{v}_2} \right) = -\omega \frac{\partial T}{\partial v_1} + N,$$  (3)
where $N$ is an undetermined multiplier which is the reaction force at the point of contact $R$. This force is directed transversely to the plane of the knife edge.

In this case it is more convenient to represent this system in the variables $(p, \omega)$, where $p$ is the momentum given by the relation

$$p = \frac{\partial T}{\partial v_1} \bigg|_{v_2=0} = m\left(\dot{v}_1 - c_2(t)\omega + \ddot{\epsilon}_1(t)\right).$$

From the last equation of (3) we find an expression for the reaction force:

$$N = \left(1 - \frac{mc_1(t)}{I_1}\right)\omega p + \left(\dot{\epsilon}_1(t) - \frac{c_1(t)}{I_1}(I(t) - mc_1(t)\dot{\epsilon}_1(t))\right)m\omega$$

$$- \frac{mc_1(t)}{I_1}(mc_2(t)\dot{\epsilon}_1(t) + \dot{k}(t)) + m\ddot{\epsilon}_2(t).$$

Finally, from (1) and (3) we obtain equations of motion in the following form:

$$\dot{p} = mc_1(t)\omega^2 + m\omega\ddot{\epsilon}_2(t),$$

$$I(t)\ddot{\omega} = -c_1(t)\omega p - (I(t) - mc_1(t)\dot{\epsilon}_1(t))\dot{\omega} - mc_2(t)\dot{\epsilon}_1(t) - \dot{k}(t),$$

$$\dot{\varphi} = \omega, \quad m\dot{x} = (p + c_2(t)\omega - \dot{\epsilon}_1(t))\cos\varphi, \quad m\dot{y} = (p + c_2(t)\omega - \dot{\epsilon}_1(t))\sin\varphi.$$

In (5), the nonautonomous reduced system governing the evolution of $(p, \omega)$ can be considered as a separate set. Of great interest is the question of whether this system has trajectories unbounded on the plane $(p, \omega)$. In this case the sleigh is observed to accelerate, that is, the kinetic energy and hence the velocity of the platform must increase indefinitely with time.

As for acceleration of the sleigh, one should distinguish between cases where the reaction force $N$ is a bounded and an unbounded function of time. Physically, unbounded increase in the reaction force $N$ implies that, at a certain instant of time, slipping will start in the direction transverse to the plane of the knife edge (i.e. the constraint (2) will be violated).

What is of interest from a practical point of view is acceleration for which the reaction force $N$ is a bounded function of time.

We consider separately several particular cases where the position of the center of mass $c(t)$, the moment of inertia $I(t)$ and the gyrostatic momentum $k(t)$ depend on time.

**Balanced case.** If the system is balanced relative to the knife edge $c_1 = 0$ (i.e. the center of mass of the system lies on the axis $Ry_1$), then the reduced system reduces to a linear one. In this case the equations of motion possess an additional integral [3]:

$$F = I(t)\omega + k(t).$$

If we fix the level set of the integral $F = f$, then the equation describing the momentum can be represented as

$$\dot{p} = \frac{mc_2(t)}{I(t)}(f - k(t)).$$

In [3], attention is given to the case in which the functions $I(t)$, $k(t)$, $c_2(t)$ periodically depend on time. Then, according to (6), the angular velocity $\omega$ is also a periodic function of time, and momentum $p$ depends periodically on time or grows linearly with time. In the latter case, acceleration is observed and, as follows from (6), the reaction force $N$ is an unbounded function of time. Moreover, as numerical experiments show, the trajectory of the point of
The contact in this case has no mean motion [3] along a certain direction\(^5\) (in what follows, we will use the term *mean motion* [48] for brevity).

**Transverse oscillations.** In [1, 3], a detailed analysis is made of the case in which the center of mass of the platform (point \(C\)) lies in the plane of the knife edge and the material point executes oscillations transverse to this plane (see figure 2). Its position in the moving coordinate system \(R_{x_1y_1}\) is defined by the radius vector

\[
\rho = (a, b \sin(\Omega t)).
\]

In this case, the reduced system can be represented as

\[
I(t) = I_s + mb^2 \mu (1 - \mu) \sin^2(\Omega t), \quad k(t) = mb \mu \Omega \cos(\Omega t),
\]

\[
c_1 = \text{const}, \quad c_2(t) = \mu b \sin(\Omega t),
\]

where \(I_s\) is the moment of inertia of the platform and \(\mu \in (0, 1)\) is the ratio of the mass of the point to that of the entire system.

In this case, the reduced system can be represented as

\[
x(t) = Pr + \chi(t), \quad y(t) = Qr + \eta(t),
\]

where \(P, Q, \sigma > 0\) are some constants and \(\chi(t)\) and \(\eta(t)\) are bounded functions of time.

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\(^5\) In this case, by mean motion we mean a change in the coordinates of the contact point of the sleigh, \((x, y)\), such that

\[
x(t) = Pr + \chi(t), \quad y(t) = Qr + \eta(t),
\]

where \(P, Q, \sigma > 0\) are some constants and \(\chi(t)\) and \(\eta(t)\) are bounded functions of time.
\[ \dot{p} = mc_1 \omega^2 + mb \mu \Omega \cos(\Omega t) \omega, \]
\[ \dot{\omega} = \frac{c_1 \omega + mb^2 \mu (1 - \mu) \sin^2(\Omega t) \omega - mba \mu \Omega^2 \sin(\Omega t)}{I - mb^2 \mu (1 - \mu) \sin^2(\Omega t)}. \]  

(8)

Consider the quadratic function [3]:
\[ F = ac_1 p^2 + \left( (I_s + mb^2 \mu (1 - \mu) \sin^2(\Omega t)) \omega + mba \mu \cos(\Omega t) \right)^2. \]

The derivative of this function along the system (8) has the form
\[ \dot{F} = -2c_1 p^2 \left( mb^2 \mu (1 - \mu) \sin^2(\Omega t) + I_s - mac_1 \right). \]

The conditions for which \( \dot{F} \) in the half-planes \( p > 0 \) and \( p < 0 \) is sign-definite are defined by the following relations:
\[ ac_1 < 0, \quad \text{or} \quad I_s - mac_1 > 0. \]  

(9)

In [3] it is noted that, for parameters satisfying (9), one always observes acceleration during which only momentum \( p \) grows indefinitely with time (numerical experiments show that it grows in proportion to \( \tau^2 \)). The first relation of (9) has a clear physical interpretation—the center of mass of the system and the oscillating point lie on different sides from the knife edge.

If relations (9) are not satisfied, then one observes chaotic oscillations and multistability for which all trajectories of the reduced system (8) are bounded. Also, on the Poincaré map one observes a strange attractor which can coexist with invariant curves.

However, the conclusion on acceleration is not rigorously proved in this case. The question of the behavior of the reaction force \( N \) remains also open. In addition, numerical experiments show that the point of contact has mean motion.

**A sleigh with a rotor.** In this paper we consider the motion of the Chaplygin sleigh only with gyrostatic momentum, that is, \( I, c_1, c_2 = \text{const} \). In this case, gyrostatic momentum can be generated, for example, by two point masses rotating about the common center (see figure 3).

Assuming \( c_1 \neq 0 \), we define dimensionless variables and parameters:
\[ \alpha = \frac{mc_1}{I}, \quad u = \frac{\tau}{\Omega}, \quad \tau = \Omega t, \quad X = \frac{x}{c_1}, \quad Y = \frac{y}{c_1}, \]
\[ A = \frac{mc_1^2}{I} \in (0, 1), \quad a = \frac{c_2}{c_1}, \quad \lambda = \frac{k}{c_1}. \]  

(10)
where \( \Omega \) is some constant that has dimension inverse to time; if the function \( k(\tau) \) is periodically changed, the constant \( \Omega \) will be the rotation frequency of the rotor. The equations of motion in terms of these variables become

\[
\begin{align*}
\frac{d\alpha}{d\tau} &= Au^2, \quad \frac{du}{d\tau} = -\alpha u - \lambda(\tau), \\
\frac{d\varphi}{d\tau} &= u, \quad \frac{dX}{d\tau} = \left(\frac{\alpha}{A} + au\right) \cos \varphi, \quad \frac{dY}{d\tau} = \left(\frac{\alpha}{A} + au\right) \sin \varphi.
\end{align*}
\]

(11)

We represent the relation for the reaction force in dimensionless variables in the form

\[
\Lambda = \frac{N}{ma\Omega^2} = \left(\frac{1}{A} - 1\right) \alpha u - \lambda(\tau).
\]

Remark 1. Let the gyrostatic momentum \( k(t) \) and the moment of inertia \( I(t) \) depend on time and let the center of mass be fixed, \( e = \text{const} \). We show that in this case the reduced system can be represented in the form (11), where \( A \) is a positive function of time.

Indeed, the reduced system has the form

\[
\dot{p} = mc_1 I(t)^2 M^2, \quad \dot{M} = -c_1 I(t) Mp - \dot{k},
\]

where \( M = I(t)\omega \).

Using the fact that the moment of inertia \( I(t) \) is a positive function, we rescale time as

\[
\tau = \Omega \int mc_1^2 dt
\]

and introduce new variables

\[
\alpha = \frac{p}{mc_1 \Omega}, \quad u = \frac{M}{mc_1^2 \Omega}.
\]

The equations of motion in this case can be represented as

\[
\begin{align*}
\frac{d\alpha}{d\tau} &= A(\tau) u^2, \quad \frac{du}{d\tau} = -\alpha u - \lambda(\tau), \\
\lambda(\tau) &= \frac{dk}{d\tau}, \quad A(\tau) = \frac{mc_1^2}{I(\tau)}.
\end{align*}
\]

2. Proof of the existence of nonlinear acceleration

Consider in more detail the question of ‘constant’ acceleration of the sleigh by means of a rotor. This question reduces to investigating the possibility of existence of unbounded trajectories for the reduced system (11). We note that parameter \( A \) in this system can be eliminated by the transformation

\[
u \rightarrow \frac{1}{\sqrt{A}} u, \quad \lambda(\tau) \rightarrow \frac{1}{\sqrt{A}} \lambda(\tau),
\]

(13)
after which the equations of motion can be represented as
\[ \alpha' = u^2, \quad u' = -\alpha u - \lambda(\tau), \] (14)
where the prime denotes the derivative with respect to \( \tau \).

The right-hand sides of the system of differential equations (14) contain quadratic terms. Solutions of such systems can go to infinity in finite time. The simplest example of a system with this property is \( x' = x^2, \ x \in \mathbb{R} \).

We show that all solutions of the system (14) are defined on the whole axis of new time \( R = \{ \tau \} \). For this purpose we consider the function
\[ V = \frac{1}{2}(u^2 + \alpha^2). \]
By virtue of (14) the derivative of this function is
\[ V' = -u\lambda(\tau). \]
Hence,
\[ |V'| = |u||\lambda| \leq \sqrt{2}\sqrt{V}|\lambda|. \]
Since \( V \geq 0 \) and \( \sqrt{V} \geq 0 \), it follows that
\[ |V^{-\frac{1}{2}}V'| \leq \sqrt{2}|\lambda|, \]
or
\[ |2(V^\frac{1}{2})'| \leq \sqrt{2}|\lambda|. \]
Integrating the inequalities
\[ -\frac{1}{\sqrt{2}}|\lambda(\tau)| \leq |(V^\frac{1}{2})'| \leq |(V^\frac{1}{2})'| \leq |(V^\frac{1}{2})'| \leq \frac{1}{\sqrt{2}}|\lambda(\tau)| \]
in the interval from 0 to \( \tau \), we find that
\[ |V^\frac{1}{2}(\tau) - V^\frac{1}{2}(0)| \leq \frac{1}{\sqrt{2}} \int_0^\tau |\lambda(s)|ds. \]
Consequently, the function \( V(\tau) \) (along with the function \( \alpha(\tau) \) and \( u(\tau) \)) can tend to \( +\infty \) only as \( |\tau| \to \infty \), which is the required result.

**Theorem 1.** Assume that the functions \( \lambda(\tau) \) and \( \lambda'(\tau) \) are bounded and the function \( \lambda(\tau) \) does not tend to zero as \( \tau \to +\infty \). If at the initial instant of time \( \alpha > 0 \), then for any initial value of the variable \( u \)

1. \( \alpha(\tau) \) tends to \( +\infty \) as \( \tau \to +\infty \),
2. \( u(\tau) \to 0 \) as \( \tau \to +\infty \),
3. the function \( \alpha(\tau)u(\tau) \) is bounded,
4. \( u'(\tau) \to 0 \) as \( \tau \to +\infty \).

In particular, under these conditions \( \alpha' \to 0 \) as \( \tau \to +\infty \) and the constraint reaction \( \Lambda \) is bounded. The conditions of theorem 1 for the function \( \lambda \) hold if \( k(\tau) \) is a periodic or (more generally) conditionally periodic nonconstant function of time. The proof is presented in appendix A.
Theorem 2. Suppose that the conditions of theorem 1 hold and there exists the average
\[
\langle \lambda^2 \rangle = \lim_{\tau \to +\infty} \frac{1}{\tau} \int_0^\tau \lambda^2(p)dp.
\] (15)
Then, as \( \tau \to +\infty \),
\[
\frac{\alpha^3(\tau)}{3} = (\langle \lambda^2 \rangle \tau + o(\tau)).
\] (16)

The average value of (15) exists if \( \lambda \) is a periodic or conditionally periodic function of time. If \( \lambda \) does not vanish, then, obviously, \( \langle \lambda^2 \rangle \) is positive. Formula (16) can be represented in the following equivalent form:
\[
\alpha = \sqrt[3]{3\langle \lambda^2 \rangle \tau + o(\tau)}.
\]

Proof of theorem 2. Since \( u'(\tau) \to 0 \) as \( \tau \to +\infty \) (conclusion 4 of theorem 1), according to the second equation of (14) we have
\[
\alpha u = -\lambda(\tau) + f(\tau),
\]
where \( f(\tau) = o(1) \). Hence,
\[
u = -\frac{\lambda + f}{\alpha}.
\]
Since \( \alpha' = u^2 \), it follows that
\[
\left[ \frac{1}{3} \alpha^3 \right]' = \lambda^2 - 2\lambda f + f^2.
\] (17)

Let \( g(\tau) \) be one of the functions
\[-2\lambda(\tau)f(\tau) \quad \text{or} \quad f^2(\tau).
\]
It is clear that \( g(\tau) \to 0 \) as \( \tau \to +\infty \). Consequently,
\[
\int_0^\tau g(s)ds = o(s).
\]
Indeed, since \( g(\tau) = o(1) \), we have
\[
\lim_{\tau \to +\infty} \frac{1}{\tau} \int_0^\tau g(s)ds = 0.
\]
Integrating (17), we obtain the required formula:
\[
\frac{\alpha^3}{3} = \langle \lambda^2 \rangle \tau + o(\tau).
\]

Theorem 3. Suppose that the conditions of theorem 1 hold, \( u(0) = 0 \) and the analytic function \( k(\tau) \) is such that for any \( a > 0 \) one can find the root of the equation
which is strictly larger than $a$. Then the function $u(\tau)$ has infinitely many zeros as $\tau \to +\infty$.

Proof of theorem 3 uses the formula

$$u(\tau) = -\frac{1}{f(\tau)} \int_{0}^{\tau} \lambda(p)f(p)dp,$$

(18)

and the property of strict monotone growth of the function $f$. The infinity of the number of zeros of the function (18) follows from the following general result [32, section 6].

**Lemma 1.** Let $\tau_1$ be the first positive zero of the analytic function

$$\int_{a}^{\tau} \lambda(t)dt$$

(19)

and let $f$ be a positive nondecreasing function. Then the integral

$$\int_{a}^{\tau} \lambda(t)f(t)dt$$

vanishes on the interval $(a, \tau_1].$

In our case, in view of (10) and (13), the integral (19) is

$$B(k(\tau) - k(a)), \quad B = \frac{1}{\sqrt{AI\Omega}},$$

and therefore the conclusion of theorem 3 follows from formula (18) and lemma 1.

**Corollary 1.** Suppose that the conditions of theorem 1 hold, $u(0) = 0$, and the analytic function $k(\tau)$ is periodic with period $T$. Then the function $u(\tau)$ has infinitely many zeros as $\tau \to +\infty$, and the distance between its neighboring zeros does not exceed $T$.

3. Dynamics in configuration space

As shown in the previous section for solutions to the reduced system, in view of the transformation (13) we have

$$\alpha(\tau) = \sigma \tau^{\frac{1}{3}} + o(\tau^{\frac{1}{3}}), \quad u(\tau) = -\frac{\lambda(\tau)}{\sigma} \tau^{-\frac{1}{3}} + o(\tau^{-\frac{1}{3}}),$$

$$\sigma = (3A(\lambda^2)\frac{1}{3}),$$

(20)

where $(\lambda^2)$ is defined by (15).

From the given solutions of the reduced system the orientation of the Chaplygin sleigh and the motion of the point of contact are defined by quadratures (12). If we restrict ourselves to the asymptotics (20), we obtain

$$\frac{d\varphi}{d\tau} = -\frac{\lambda(\tau)}{\sigma} \tau^{-\frac{1}{3}},$$

$$\frac{dX}{d\tau} = \frac{2}{3} \tau^{\frac{1}{3}} \cos \varphi, \quad \frac{dY}{d\tau} = \frac{2}{3} \tau^{\frac{1}{3}} \sin \varphi.$$
We let $\tau = \tau_0 > 0$ denote time from which the asymptotics (20) describes ‘well’ the solution of the reduced system, and supplement the system (21) with the following initial conditions:

$$\varphi(\tau_0) = X_0, \quad X(\tau_0) = Y_0, \quad Y(\tau_0) = Z_0.$$  

(22)

Consider the system (21) in more detail in two cases:

1. **Constantly accelerating rotor** — the angular velocity of the rotor is a linear function of time, in this case $\lambda(\tau) = \lambda_0 = \text{const}$;

2. **Periodically oscillating rotor** — the angular velocity of the rotor is a periodic function of time. As an example we consider the function $\lambda(\tau) = \lambda_0 \sin \tau$.

**Constantly accelerating rotor** ($\lambda(\tau) = \lambda_0 > 0$). Explicitly integrating the system (21), we find

$$\varphi(\tau) = -s_1(\tau^\frac{3}{2} - \tau_0^\frac{3}{2}) + \varphi_0, \quad s_1 = \left(\frac{9\lambda_0}{8A}\right)^\frac{1}{3},$$

$$X(\tau) = 2\left(\cos \varphi(\tau) - s_1 \tau^\frac{3}{2} \sin \varphi(\tau)\right) - 2\left(\cos \varphi_0 - s_1 \tau_0^\frac{3}{2} \sin \varphi_0\right) + X_0,$$

$$Y(\tau) = 2\left(\sin \varphi(\tau) + s_1 \tau^\frac{3}{2} \cos \varphi(\tau)\right) - 2\left(\sin \varphi_0 + s_1 \tau_0^\frac{3}{2} \cos \varphi_0\right) + Y_0.$$  

(23)

(24)

As we see, the trajectory of the point of contact is an untwisting spiral (figure 4) and in this case there is no mean motion of the sleigh.

**Periodically oscillating rotor** ($\lambda(\tau) = \lambda_0 \sin \tau$). A typical behavior of the solutions to the system (11) and (12) in this case is presented in figure 5. As numerical experiments show, the following statement holds.

In the case of a periodically oscillating rotor the angle of rotation of the sleigh $\varphi$ tends to a finite limit, and the ‘limit’ motions of the point of contact are oscillations (with constant amplitude) in a neighborhood of a straight line.

In order to show the validity of this statement, we consider the system (21) in the approximation (20) and make use of the relation [42, p 401]
This yields

\[
\varphi(\tau) = \tilde{\varphi} + s_2 \frac{\cos \tau}{\tau^\frac{2}{3}} + O(\tau^{-\frac{4}{3}}) ,
\]

that is, the angle of rotation of the sleigh tends to a fixed value of \( \tilde{\varphi} \) as \( t \to +\infty \). We substitute the function \( \varphi(\tau) \) obtained into the equations for the evolution of the point of contact and expand the resulting expressions in powers of \( \tau^{-\frac{2}{3}} \). Neglecting terms of order \( O(\tau^{-\frac{4}{3}}) \) and \( O(\tau_0^{-\frac{4}{3}}) \), we represent the equations of motion for the point of contact in the form.
\[
\frac{d\tilde{X}}{d\tau} = \frac{A}{\lambda} \tau^\dagger \cos \tilde{\varphi} + s_2 \cos \tau \sin \tilde{\varphi},
\]
\[
\frac{d\tilde{Y}}{d\tau} = \frac{A}{\lambda} \tau^\dagger \sin \tilde{\varphi} - s_2 \cos \tau \cos \tilde{\varphi}.
\]

If we rotate the fixed coordinate system through angle \(\tilde{\varphi}\):

\[
\tilde{X} = X \cos \tilde{\varphi} + Y \sin \tilde{\varphi}, \quad \tilde{Y} = -X \sin \tilde{\varphi} + Y \cos \tilde{\varphi},
\]

then the equations of motion become

\[
\frac{d\tilde{X}}{d\tau} = \frac{A}{\lambda} \tau^\dagger, \quad \frac{d\tilde{Y}}{d\tau} = -s_2 \frac{A}{\lambda} \cos \tau.
\]

Explicitly integrating this system, we find

\[
\tilde{X}(\tau) = \tilde{X}_0 + \frac{A}{\lambda} \tau^\dagger - \tau_0^\dagger, \quad \tilde{Y}(\tau) = \tilde{Y}_0 - s_2 \frac{A}{\lambda} (\sin \tau - \sin \tau_0),
\]

where \(\tilde{X}_0\) and \(\tilde{Y}_0\) are initial conditions calculated using (22). Thus, along the axis \(\tilde{X}\) the sleigh moves away in proportion to \(\tau^\dagger\), and along the axis \(\tilde{Y}\) it executes oscillations with constant amplitude \(\frac{A}{\lambda} = \lambda_0\).

Let us compare relation (25) with the numerical solution of the system (11)–(12). To do so, we calculate the angle:

\[
\tilde{\varphi}_n = \frac{\psi_1 + \psi_2}{2}, \quad \psi_1 = \frac{1}{2\pi} \int_{\tau_0}^{\tau_0 + 2\pi n} \varphi(\tau) d\tau,
\]
\[
\psi_2 = \frac{1}{2\pi} \int_{\tau_0}^{\tau_0 + \pi (2n+1)} \varphi(\tau) d\tau, \quad n \in \mathbb{N},
\]

where the function \(\varphi(\tau)\) is a numerical solution of the initial system. The angle of rotation is defined by

\[
\tilde{\varphi} = \lim_{n \to \infty} \varphi_n,
\]

that is, the larger \(n\), the more exactly \(\varphi_n\) approximates \(\tilde{\varphi}\).

A typical view of the trajectory of the point of contact is shown in figure 6. Also, the dependence \(\tilde{X}(\tau)\) is plotted numerically in figure 7. As can be seen, the relations obtained for the amplitude, as well as the growth rate of \(\tilde{X}\), agree well with numerical experiments. However, a rigorous proof of this fact requires investigating more detailed estimates for the functions \(\alpha(\tau)\) and \(u(\tau)\) (than that considered in this section (20)).

4. Viscous friction

We assume that the motion of the sleigh occurs in the presence of viscous friction force with a dissipative Rayleigh function of the form

\[
R = \frac{1}{2} (\nu_1 \dot{\varphi}_1^2 + \nu_2 \omega^2).
\]
For the system of Lagrange equations with undetermined multipliers we write
\[
d\frac{d}{dt} \left( \frac{\partial T}{\partial v_1} \right) = \omega \frac{\partial T}{\partial v_2} - \nu_1 v_1, \quad \frac{d}{dt} \left( \frac{\partial T}{\partial v_2} \right) = v_2 \frac{\partial T}{\partial v_1} - v_1 \frac{\partial T}{\partial v_2} - \nu_2 \omega,
\]
\[
d\frac{d}{dt} \left( \frac{\partial T}{\partial \omega} \right) = -\omega \frac{\partial T}{\partial \sigma_1} + N.
\]

In the presence of friction force the reduced system can be brought, after a transformation similar to (13) under the assumption \( c_2 = 0 \), to the form
\[
\alpha' = Au^2 - \sigma_1 \alpha, \quad u' = -\alpha u - \lambda(\tau) - \sigma_2 u,
\]
\[
\frac{d\varphi}{d\tau} = u, \quad \frac{dX}{d\tau} = \left( \frac{\alpha}{\lambda} + au \right) \cos \varphi, \quad \frac{dY}{d\tau} = \left( \frac{\alpha}{\lambda} + au \right) \sin \varphi,
\]
where \( \sigma_1 \) and \( \sigma_2 \) are the coefficients of friction. As numerical experiments show (see figure 8), there is no acceleration in (27) in this case.

If \( \nu_1 = 0 \) (\( \sigma_1 = 0 \)), but \( \nu_2 \neq 0 \), then, by making the change of variable

\( \tau' = \tau \quad \alpha' = \alpha + \sigma_2 u \),

\( \varphi' = \varphi + \lambda(\tau) \),

\( u' = u - \sigma_2 u \),

the system becomes
\[
\frac{d\varphi'}{d\tau'} = u', \quad \frac{dX'}{d\tau'} = \left( \frac{\alpha}{\lambda} + au \right) \cos \varphi', \quad \frac{dY'}{d\tau'} = \left( \frac{\alpha}{\lambda} + au \right) \sin \varphi',
\]
where \( \sigma_1 \) and \( \sigma_2 \) are the coefficients of friction.
one can reduce the system (27) to the system (11), consequently, acceleration is observed in this case.

Let us perform normalizing transformations of the variables, functions and time

$$\begin{align*}
\alpha &\rightarrow \alpha + \sigma_2 \\
\lambda &\rightarrow \frac{\sigma_1^2}{\sqrt{A}} \lambda, \quad \sigma_2 \rightarrow \sigma_1 \mu, \quad \sigma_1 \, d\tau \rightarrow d\tau,
\end{align*}$$

Figure 8. A typical view of a periodic solution and of trajectories tending to it for the system (28) with $\lambda(\tau) = \sin \tau$, $\mu = 0.3$.

Figure 9. The region $D_{\mu,\lambda}$ on the plane $(\alpha, u)$. 

Let us perform normalizing transformations of the variables, functions and time

$$\begin{align*}
\alpha &\rightarrow \sigma_1 \alpha, \quad u \rightarrow \frac{\sigma_1 \mu}{\sqrt{A}} \\
\lambda &\rightarrow \frac{\sigma_1^2}{\sqrt{A}} \lambda, \quad \sigma_2 \rightarrow \sigma_1 \mu, \quad \sigma_1 \, d\tau \rightarrow d\tau,
\end{align*}$$
and represent the system as
\[
\alpha' = u^2 - \alpha, \quad u' = -\alpha u - \lambda(\tau) - \mu u. \tag{28}
\]

We now apply the Brouwer’s fixed point theorem to this system (see appendix B). To do so, we consider a closed region \(D_{\mu, \lambda_m}\) on the plane \((\alpha, u)\) bounded by four straight lines (see figure 9)
\[
\begin{align*}
\gamma_1 : \alpha &= 0, & \gamma_2 : \alpha &= \sqrt{\frac{\lambda_m}{\mu}}, \\
\gamma_3 : u &= \lambda_m, & \gamma_4 : u &= -\lambda_m.
\end{align*}
\tag{29}
\]
where \(\lambda_m = \max |\lambda(\tau)|\).

We show that this region is invariant under the flow. Indeed, according to the second equation in (28), the vector field on the curves \(\gamma_3\) and \(\gamma_4\) is directed into the region \(D_{\mu, \lambda_m}\) (since the inequalities \(\dot{u} \leq 0\) and \(\dot{u} \geq 0\) are satisfied for all \(t\) on the upper straight line \(\gamma_3\) and on the lower straight line \(\gamma_4\), respectively). Similarly, on the straight line \(\gamma_1\) the vector field is also directed into the region (since on this straight line \(\dot{\alpha} \geq 0\)), on the other hand, on the straight line \(\gamma_2\) the vector field is also directed inside (since the segment under consideration gets into the parabola \(\alpha = u^2\), where \(\dot{\alpha} \leq 0\)).

Hence, on the basis of theorem B.1 of appendix B we obtain the following theorem.

**Theorem 4.** *In the system (28) with \(\mu \neq 0\) there exists at least one periodic solution in the region \(D_{\mu, \lambda_m}\).*

Now, in order to clarify the conditions for uniqueness of the periodic solution in the region \(D_{\mu, \lambda_m}\), we find a region on the plane \((\alpha, u)\) where the map for a period of the system (28) is compressing. According to appendix B, this requires ascertaining the region of negative definiteness of the symmetric part of the Jacobian of the right-hand side of the system (28):
\[
B = \begin{pmatrix}
-1 & \frac{\mu}{2} \\
\frac{\mu}{2} & -\alpha - \mu
\end{pmatrix}.
\]
Its eigenvalues \(b_{\pm}\) are given by
\[
b_{\pm} = -\frac{1}{2} (\alpha + \mu + 1 \pm \sqrt{\alpha^2 + (\alpha + \mu - 1)^2}).
\]

We thus find that the ‘compression region’ (i.e. the region where \(b_{\pm} < 0\)) is given by
\[
4(\alpha + \mu) \geq u^2.
\]

Using the equations of the upper and lower boundaries (29) of the region \(D_{\mu, \lambda_m}\), we obtain

**Theorem 5.** *If in the system (28)
\[
2 \mu^{3/2} \geq \lambda_m,
\tag{30}
\]
then in the region \(D_{\mu, \lambda_m}\) there exists the only periodic orbit to which all other trajectories in \(D_{\mu, \lambda_m}\) tend exponentially.*

We note that in this section we obtain an estimate of the region of existence of the limit cycle \(D_{\mu, \lambda_m}\) from the point of view of its localization. Generally speaking, one can pose the problem of more exact localization of the limit cycle, that is, the problem of finding the smallest possible region \(D' \subset D_{\mu, \lambda_m}\) containing this cycle. On the other hand, under the conditions of theorem 2 one can also pose the problem of finding the largest possible region \(D'' \supset D_{\mu, \lambda_m}\) in which all trajectories tend to the only limit cycle.
The system (27) in the case $\sigma_2 = 0$ and $\lambda(\tau) = \sin \tau$ was considered in [21]. In this case, inequality (30) does not hold. However, numerical experiments show that theorem 2 remains valid.

**Dynamics in configuration space.** A typical behavior of solutions to the system (27) is shown in figure 10. We see that the ‘limit’ motions of the point of contact, as in the absence of friction force, are oscillations (with constant amplitude) in a neighborhood of a straight line. In this case, in a neighborhood of the limit cycle the rotation angle and the translational and angular velocities of the sleigh change periodically with time (i.e. there is no unbounded increase in the translational velocity).

The trajectory of the point of contact in the variables of section 4 is presented in figure 11. This implies that, as the friction coefficient increases, the oscillation amplitude decreases.

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Since the angle $\varphi(\tau)$ in a neighborhood of the limit cycle changes periodically with time, it suffices to set $n = 1$ in (26).
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Appendix A. Proof of theorem 1

Conclusion 1 is proved using the following lemma.

Lemma A.1 (Hadamard [25]). If the function \( f(\tau) \) tends to the limit as \( \tau \rightarrow +\infty \) and the functions \( f'(\tau) \) and \( f''(\tau) \) are bounded, then \( f'(\tau) \rightarrow 0 \) as \( \tau \rightarrow +\infty \).

The monotone growth of the function \( \alpha(\tau) \) follows from the fact that the derivative \( \alpha' \) is positive for \( u \neq 0 \) and that \( u = 0 \) is not an invariant submanifold of the system (14).

Assume that the function \( \alpha(\tau) \) is bounded. Then

\[
\lim_{\tau \rightarrow +\infty} \alpha(\tau) = \bar{\alpha} > 0.
\]

The positiveness of the limit follows from the assumption that \( \alpha(0) > 0 \).

We show that in this case the function \( u(\tau) \) is bounded. Indeed, it satisfies the linear differential equation

Figure 11. Trajectory of the point of contact plotted numerically for fixed parameters \( \lambda_0 = 1, A = 1, \sigma_2 = 0 \) and initial conditions \( \tau = 0, \alpha = 0, u = 0, \varphi = 0, X = 0, Y = 0 \) for different \( \sigma_1 \).
We solve it by the method of variation of constants. The solution of the homogeneous equation is
\[ u = Ce^{-\int_0^\tau \alpha(s)ds}. \]

Now, assuming \( C \) to be a function of \( \tau \), we obtain
\[ C' = -\lambda(\tau)e^{\int_0^\tau \alpha(s)ds}. \]

Let us introduce a new function \( \tilde{C} \) by the following formula:
\[ \tilde{C}' = -\mu e^{\int_0^\tau \alpha(s)ds}, \quad \mu = \sup |\lambda(\tau)|. \]

It is clear that\[ |\tilde{C}'| \geq |C'|. \] Consequently, if \( C(0) = \tilde{C}(0) \), then
\[ |C(\tau)| \leq |\tilde{C}(\tau)| \quad (A.2) \]
for all \( \tau \geq 0 \).

Let us calculate
\[ \lim_{\tau \to +\infty} \frac{\tilde{C}(\tau)e^{\int_0^\tau \alpha(s)ds}}{e^{\int_0^\tau \alpha(s)ds}} = \lim_{\tau \to +\infty} \frac{\tilde{C}(\tau)}{e^{\int_0^\tau \alpha(s)ds}}. \]

Since \( \alpha(s) \to \check{\alpha} > 0 \), the denominator of this fraction tends to \(+\infty\) (as does the absolute value of the numerator). Therefore, one can use L’Hôpital’s rule: this limit is
\[ \lim_{\tau \to +\infty} \tilde{C}' = -\frac{\mu}{\check{\alpha}}. \]

Thus (according to (A.2)), the function \( |u(\tau)| = |C(\tau)e^{\int_0^\tau \alpha(s)ds}| \) is bounded.

Further, according to (14),
\[ \alpha'' = 2u\alpha' = -2u\alpha^2 - 2u\lambda(\tau) \]
is also bounded. Therefore (by Hadamard’s lemma), \( \alpha'(\tau) \to 0 \). But then (according to (14)) \( u(\tau) \to 0 \) as \( \tau \to +\infty \).

We now consider the second derivative
\[ u'' = -\alpha' u - \alpha u' + \lambda'(\tau). \]

This derivative is bounded since (according to (14)) \( u' \) is bounded. Since \( u(\tau) \to 0 \) and the derivatives \( u'(\tau) \) and \( u''(\tau) \) are bounded, again according to Hadamard’s lemma, \( u''(\tau) \to 0 \) as \( \tau \to +\infty \). But this contradicts the second equation of (14), since \( \alpha(\tau) \to \check{\alpha} \) (by assumption), \( u(\tau) \to 0 \), and the function \( \lambda(\tau) \) does not tend to zero as \( \tau \to +\infty \).

The resulting contradiction proves conclusion 1. Next, the following lemma is needed.

**Lemma A.2.** *Under the conditions of theorem 1 the relation*
\[ \frac{\int_0^\tau f(p)dp}{f(\tau)}, \quad \text{where} \quad f(p) = e^{\int_0^p \alpha(s)ds}, \quad (A.3) \]
*tends to zero as \( \tau \to +\infty \).*
Proof. Since the denominator of (A.3) tends to infinity as \( \tau \to +\infty \), one can use L’Hôpital’s rule and the statement (already proved) that \( \alpha(\tau) \to +\infty \) as \( \tau \to +\infty \).

We now prove conclusion 2. The function \( u(\tau) \) as a solution of the linear inhomogeneous differential equation

\[
u' = -\alpha(\tau)u - \lambda(\tau)
\]

is the sum of two functions

\[
u(0)e^{-\int_0^\tau \alpha(s)ds}
\]

and

\[-e^{-\int_0^\tau \alpha(s)ds} \int_0^\tau \lambda(p)e^{\int_0^p \alpha(s)ds}dp.\]

The function (A.4) tends superexponentially fast to zero as \( \tau \to +\infty \). Since \( \lambda(\tau) \) is bounded, the function (A.5) also tends to zero according to lemma A.2.

To prove conclusion 3, we make use of formulae (A.4) and (A.5), the sum of which is the function \( u(\tau) \). The product of \( \alpha(\tau) \) and the function (A.4) tends to zero as \( \tau \to +\infty \). Indeed, according to L’Hôpital’s rule, the limit of this product is equal to

\[
\frac{\alpha(\tau)\int_0^\tau f(p)dp}{f(\tau)} \text{, where } f(p) = e^{\int_0^p \alpha(s)ds}.
\]

According to L’Hôpital’s rule, the limit of this function as \( \tau \to +\infty \) is equal to

\[
\lim_{\tau \to +\infty} \frac{\alpha(\tau)\int_0^\tau f(p)dp + \alpha(\tau)f(\tau)}{\alpha(\tau)f(\tau)} = 1
\]

(by lemma A.2). Consequently, the product \( \alpha(\tau)u(\tau) \) is indeed bounded.

According to the second equation of (14), the derivative \( u'(\tau) \) is bounded. We prove that \( u'(\tau) \to 0 \) as \( \tau \to +\infty \).

Indeed, according to (A.4) and (A.5),

\[
u' = \alpha u + \lambda = \frac{\alpha(u(0) - \int_0^\tau \lambda(p)f(p)dp) + \lambda(\tau)f(\tau)}{f(\tau)}.
\]

where \( f \) is given by formula (A.3). It is clear that

\[
u(0)\frac{\alpha(\tau)}{f(\tau)} \to 0
\]

as \( \tau \to +\infty \). We calculate the limit of the two remaining terms in (A.6) again by L’Hôpital’s rule. It is equal to
\[
\lim_{\tau \to +\infty} -\alpha' \int_{0}^{\tau} \lambda f dp + \lambda f.
\]  
(A.7)

Taking into account the assumption about boundedness of the functions \(\lambda(\tau)\) and \(\lambda'(\tau)\), as well as the already established properties \(\alpha(\tau) \to +\infty\), \(\alpha'(\tau) \to 0\) and lemma A.2, we find that the limit (A.7) is zero. This proves conclusion 4.

**Appendix B. The principle of compressing maps**

1. In this section, we briefly formulate in a form convenient for our purposes some results on the periodic solutions of the system which depend periodically on time. Let the following system be given in the Euclidean space \(E^n = \{x = (x_1, \ldots, x_n)\}:
\[
\dot{x} = v(x, t), \quad v(x, t + T) = v(x, t).
\] (B.1)

We first define the notion of an invariant subset \(D \subset M\).

**Definition B.1.** If for all initial conditions \(t_0 \in [0, T]\) and \(x_0 \subset D\) the trajectories \(x(t) \subset D, t > t_0\), then \(D\) is said to be an invariant subset.

If the vector field (B.1) has no singular points inside \(D\), then a natural family of maps for a period is defined:
\[
\Pi_{t_0} : D \to D,
\] (B.2)

which assigns to each point \(x_0\) point \(x\) at time \(t_0 + T\), on the trajectory of the system with initial conditions \(x(t_0) = x_0\). The existence and uniqueness theorem and the theorem of continuous dependence on initial conditions guarantee that the maps \(\Pi_{t_0}\) are continuous and biunique on their image.

If the set \(D\) is homeomorphic to a closed ball, then, according to the Brouwer’s fixed point theorem, any map (B.2) has a fixed point inside \(D\):
\[
\Pi_{t_0}(x^*) = x^*.
\]

In the flow (B.1) this fixed point corresponds to the periodic solution
\[
x^*_p(t) = x^*_p(t + T), \quad x^*_p(t_0) = x^*.
\]

Thus, the following theorem holds.

**Theorem B.1.** Suppose that the system (B.1) admits an invariant set \(D\) that is homeomorphic to a closed ball and does not contain any singular points of the vector field. Then in \(D\) there is at least one periodic solution.

2. Now assume that the flow (B.1) admits a closed invariant set \(D\) inside which it possesses the property of uniform compression, namely: the following inequality is satisfied for any two trajectories \(x_1(t), x_2(t)\) inside \(D\) at all instants of time \(t\):
\[
|x_1 - x_2| = \frac{|x_1 - x_2, v(x_1, t) - v(x_2, t)|}{|x_1 - x_2|} \leq -h |x_1 - x_2|,
\] (B.3)

where \(h > 0\) is some constant.
In this case the maps (B.2) are also compressing
\[ |\Pi_0(x_1) - \Pi_0(x_2)| \leq \lambda|x_1 - x_2|, \quad \lambda = e^{-hT} < 1. \tag{B.4} \]

Indeed, let \( \Delta(t) = |x^{(1)}(t) - x^{(2)}(t)| \), where \( x^{(k)}(t), \ k = 1, 2 \) are trajectories of the system (B.1) that satisfy the initial conditions \( x^{(k)}(t_0) = x_k, \ k = 1, 2 \). Then \( \Delta(t_0) = |x_1 - x_2| \) and \( \Delta(t_0 + T) = |\Pi_0(x_1) - \Pi_0(x_2)| \). From the above inequality (B.3) and the condition \( \Delta(t) > 0 \) we obtain
\[ (\ln \Delta(t))' \leq -h. \]

Integrating this relation for a period, we obtain (B.4).

Thus, we apply to \( \Pi_0 \) the principle of compressing Banach maps [51], according to which \( \Pi_0 \) has a unique fixed point in \( \mathcal{D} \) to which the trajectory of any other point tends exponentially. Thus, the following theorem holds.

**Theorem B.2.** Suppose that the system (B.1) possesses the property of uniform compression inside some closed invariant set \( \mathcal{D} \). Then in \( \mathcal{D} \) there exists a unique periodic solution and all other trajectories in \( \mathcal{D} \) tend to it exponentially.

We now give the simplest criterion for uniform compression as presented in [20]. Define the Jacobian of the right-hand side of (B.1):
\[ J(x, t) = \left\| \frac{\partial v_i(x, t)}{\partial x_j} \right\|. \]

**Proposition B.1 ([20]).** Suppose that the quadratic form given by the matrix \( J \) is uniformly negative definite for all \( x \in D \) and \( t \):
\[ \langle x, J(x, t)x \rangle \leq -h(x, x), \]
where \( h > 0 \) is some constant. Then the flow of the system possesses the property of uniform compression.

Finally, we obtain the following result.

**Theorem B.3.** If the eigenvalues of the matrix
\[ B(x, t) = \frac{1}{2} (J(x, t) + J^T(x, t)), \quad x \in \mathcal{D} \]
are negative and separated from zero, then the system (B.1) possesses a unique limit cycle in \( \mathcal{D} \) to which all other trajectories tend exponentially.

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