Dynamical Phenomena Occurring due to Phase Volume Compression in Nonholonomic Model of the Rattleback

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Abstract—We study numerically the dynamics of the rattleback, a rigid body with a convex surface on a rough horizontal plane, in dependence on the parameters, applying methods used earlier for treatment of dissipative dynamical systems, and adapted here for the nonholonomic model. Charts of dynamical regimes on the parameter plane of the total mechanical energy and the angle between the geometric and dynamic principal axes of the rigid body are presented. Characteristic structures in the parameter space, previously observed only for dissipative systems, are revealed. A method for calculating the full spectrum of Lyapunov exponents is developed and implemented. Analysis of the Lyapunov exponents of the nonholonomic model reveals two classes of chaotic regimes. For the model reduced to a 3D map, the first one corresponds to a strange attractor with one positive and two negative Lyapunov exponents, and the second to the chaotic dynamics of quasi-conservative type, when positive and negative Lyapunov exponents are close in magnitude, and the remaining exponent is close to zero. The transition to chaos through a sequence of period-doubling bifurcations relating to the Feigenbaum universality class is illustrated. Several examples of strange attractors are considered in detail. In particular, phase portraits as well as the Lyapunov exponents, the Fourier spectra, and fractal dimensions are presented.

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1. INTRODUCTION

Two classes of dynamical systems are distinguished traditionally: the conservative and the dissipative ones. In physics, the term “conservative” means conservation of energy associated with the dynamic variables taken into account. For example, mechanical systems without friction, which allow the description by Lagrange or Hamilton formalism relate to the class of conservative systems [1, 2]. In the presence of friction they become dissipative systems [2]. Instead of a single system, let us think of an ensemble of objects identical to this system and differing only in initial conditions. In the state space (phase space) it corresponds to a cloud of representative points, which evolves in time, undergoing some transformations according to the motion of individual “particles” governed by the evolution rule of the system under consideration. If each part of the cloud keeps its volume constant (at least in appropriately chosen variables), it

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is the case of a conservative system. For a dissipative system any element of the cloud reduces the volume during the evolution in time, and the cloud in the state space settles down onto a subset, called the attractor (or, perhaps, on several attractors).

On the other hand, in mechanics, beside the Lagrangian and Hamiltonian systems, a class of systems with nonholonomic constraints is introduced, or, briefly, the nonholonomic systems [3–6] (The term was coined by Heinrich Hertz in the 19th century.) This class contains many problems of great practical significance, for example, relating to mobile and aircraft dynamics, robotic technology. The history of studies of these systems is rich in dramatic moments, including mistakes of eminent researchers, corrected later by further careful analysis.

In general, the class of nonholonomic systems is extremely wide. In fact, there is a hierarchy of dynamic behaviors from integrable to non-integrable ones, depending on a number of the inherent invariants and symmetries, of the presence or absence of invariant measure (in the sense of the Liouville theorem) [3–6].

An extreme non-integrable example is the dynamics of the rattleback, or Celtic stone [3–9]. The problem consists in studying motions of a rigid body with a smooth convex surface on a rough plane, when geometric and dynamical principal axes of the body do not coincide. Friction is present, but it is not capable of producing mechanical work and changing the mechanical energy of the motion. Remarkably, this and similar nonholonomic systems occupy, so to say, an intermediate place between conservative and dissipative systems in the traditional interpretation. Although the mechanical energy is conserved, the property of phase volume conservation is missing: an element of volume in the process of evolution over time may undergo compression in some regions of the state space, and extension in others. Although the motion occurs on a hypersurface of constant energy in the phase space, a variety of phenomena specific for dissipative dynamics, such as attractive stable fixed points or limit cycles is possible [4]. A significant advance was made in [5], where the occurrence of chaotic attractors in nonholonomic systems, like the rattleback, was outlined.

It should be emphasized that each attracting set in the phase space of a nonholonomic system has a symmetric partner — an object composed of orbits with behavior of the same nature, but in reversed time. An alternative kind of dynamics corresponds to invariant sets symmetric with respect to the time reversal; it is called “mixed dynamics”; chaos observed in such a case cannot be associated with “strange attractors”, at least, in the usual sense [8, 9].

Naturally, a research program arises aimed at identifying and classifying the types of dynamic behavior of nonholonomic models, in particular, establishing feasibility for the entire zoo of phenomena intrinsic to dissipative systems [11–14]. It would be interesting to find examples of different specific types of bifurcations, multistability, synchronization, quasiperiodic attractors (tori), strange attractors (chaotic and nonchaotic), hyperbolic chaos. In this context, in studying the dynamics of nonholonomic systems, such as models of the rattleback, it makes sense to turn to computer techniques developed extensively for dissipative systems, including computation of Lyapunov exponents, estimates of dimensions of attractors, drawing “bifurcation trees”, spectral analysis. Since it is necessary to study regimes and bifurcations depending on many parameters, it is appropriate to apply graphical representations of the parameter space by charts of dynamical regimes, charts of Lyapunov exponents, charts of rotation indices etc., see e.g. [13–16]. For this class of systems one of basic parameters, on which the dynamics depends significantly, will be the value of mechanical energy attributed to each regime of motion.

2. THE RATTLEBACK: FORMULATION OF THE PROBLEM

Consider a rigid body of unit mass with a smooth surface of convex shape, moving on a rough horizontal plane (Fig. 1). It is supposed that there is a single contact point of the body with the plane at any moment, and the velocity at this point is zero. It is assumed that in the region where the contact point can be placed, the surface of the body has the form of an elliptic paraboloid, with two principal radii of curvature \( a_1 \) and \( a_2 \). The distance between the center of mass and the top of the paraboloid is \( h \). The gravitational acceleration is \( g_0 \).

Equations of motion are formulated for components of the angular momentum \( \mathbf{M} = (M_1, M_2, M_3) \) and for components of the vector \( \mathbf{\gamma} = (\gamma_1, \gamma_2, \gamma_3) \) orthogonal to the plane, which
where the derivatives are expressed as

\[ M_1 = M_2 \omega_3 - M_3 \omega_2 + \omega_1 \rho - r_1 \omega_r + g_0 (r_2 \gamma_3 - \gamma_2 r_2), \]
\[ M_2 = M_3 \omega_1 - M_1 \omega_3 + \omega_2 \rho - r_2 \omega_r + g_0 (r_3 \gamma_1 - \gamma_1 r_3), \]
\[ M_3 = M_1 \omega_2 - M_2 \omega_1 + \omega_3 \rho - r_3 \omega_r + g_0 (r_1 \gamma_2 - \gamma_2 r_1), \]

(2.1)

all are defined in the coordinate frame fixed with respect to the moving rigid body. Taking into account zero velocity at the contact point, the dynamical equations can be written as [4–6, 9]

\[ \dot{r}_1 = -a_1 \gamma_1 / \gamma_3, \quad r_2 = -a_2 \gamma_2 / \gamma_3, \quad r_3 = -h + \frac{1}{2} (a_1 \gamma_1^2 + a_2 \gamma_2^2) / \gamma_3^2 \]

(2.2)

represent coordinates of the mass center relative to the point of contact. The components of the angular velocity \( \omega = (\omega_1, \omega_2, \omega_3) \) are expressed via the components of \( M \) and \( r = (r_1, r_2, r_3) \) by a set of linear algebraic equations

\[
\begin{pmatrix}
M_1 \\
M_2 \\
M_3
\end{pmatrix} =
\begin{pmatrix}
I_1 \cos^2 \delta + I_2 \sin^2 \delta + r_2^2 + r_3^2 & (I_1 - I_2) \cos \delta \sin \delta - r_1 r_2 & -r_1 r_3 \\
(I_1 - I_2) \cos \delta \sin \delta - r_2 r_1 & I_1 \sin^2 \delta + I_2 \cos^2 \delta + r_3^2 + r_1^2 & -r_2 r_3 \\
-r_3 r_1 & -r_3 r_2 & I_3 + r_1^2 + r_2^2
\end{pmatrix}
\begin{pmatrix}
\omega_1 \\
\omega_2 \\
\omega_3
\end{pmatrix}.
\]

(2.3)

Here \( I_1, I_2, I_3 \) are principal moments of inertia, and \( \delta \) defines the angle between the dynamic and geometric principal axes. In addition, the right-hand sides contain the value

\[ \rho = r_1 \dot{r}_1 + r_2 \dot{r}_2 + r_3 \dot{r}_3 \]

(2.4)

where the derivatives are expressed as

\[ \dot{r}_1 = -a_1 [(\gamma_2 \omega_3 - \gamma_3 \omega_2) \gamma_3 - (\gamma_1 \omega_2 - \gamma_2 \omega_1) \gamma_1] / \gamma_3^2, \]
\[ \dot{r}_2 = -a_2 [(\gamma_3 \omega_1 - \gamma_1 \omega_3) \gamma_3 - (\gamma_1 \omega_2 - \gamma_2 \omega_1) \gamma_2] / \gamma_3^2, \]
\[ \dot{r}_3 = [a_1 \gamma_1 (\gamma_2 \omega_3 - \gamma_3 \omega_2) + a_2 \gamma_2 (\gamma_3 \omega_1 - \gamma_1 \omega_3)] / \gamma_3^2 - (a_1 \gamma_1^2 + a_2 \gamma_2^2) (\gamma_1 \omega_2 - \gamma_2 \omega_1) / \gamma_3^2. \]

(2.5)

The system possesses the energy integral

\[ E = \frac{1}{2} (M_1 \omega_1 + M_2 \omega_2 + M_3 \omega_3) - g_0 (r_1 \gamma_1 + r_2 \gamma_2 + r_3 \gamma_3). \]

(2.6)
and the geometric integral
\[ \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1. \] (2.7)

In the 6D state space \((M_1, M_2, M_3, \gamma_1, \gamma_2, \gamma_3)\) the dynamical evolution takes place on a 4D hypersurface defined by Eqs. (2.6) and (2.7). Using the Poincaré section [3–6, 8–10], we reduce the problem to a 3D map. (Note that the phenomenology of dynamics and bifurcations for 3D maps is a subject of active research, see e.g. [8, 17].)

Equations (2.1) are invariant relative to the time-reversal symmetry
\[ t \leftrightarrow -t, \quad M_1 \leftrightarrow -M_1, \quad M_2 \leftrightarrow -M_2, \quad M_3 \leftrightarrow -M_3, \quad \gamma_1 \leftrightarrow \gamma_1, \quad \gamma_2 \leftrightarrow \gamma_2, \quad \gamma_3 \leftrightarrow \gamma_3, \] (2.8)
and to the symmetry with respect to the change of variables
\[ t \leftrightarrow t, \quad M_1 \leftrightarrow -M_1, \quad M_2 \leftrightarrow -M_2, \quad M_3 \leftrightarrow M_3, \quad \gamma_1 \leftrightarrow -\gamma_1, \quad \gamma_2 \leftrightarrow -\gamma_2, \quad \gamma_3 \leftrightarrow \gamma_3. \] (2.9)

To study the dynamical behavior of various types and to represent results graphically, it is convenient to apply the so-called Andoyer–Deprit variables [4–6, 9]
\[ L = M_3, \]
\[ G = \sqrt{M_1^2 + M_2^2 + M_3^2}, \]
\[ l = \text{arg}(M_2 + iM_1), \]
\[ H = M_1\gamma_1 + M_2\gamma_2 + M_3\gamma_3 \]
\[ g = \text{arg}[HL/G - G\gamma_3 + i(M_2\gamma_1 - M_1\gamma_2)]. \] (2.10)

The backward change of variables is determined as
\[ M_1 = \sqrt{G^2 - L^2}\sin l, \]
\[ M_2 = \sqrt{G^2 - L^2}\cos l, \]
\[ M_3 = L, \]
\[ \gamma_1 = \left[ (H/L)\sqrt{1 - L^2/G^2} + (L/G)\sqrt{1 - H^2/G^2} \cos g \right] \sin l + \sqrt{1 - H^2/G^2} \sin g \cos l, \]
\[ \gamma_2 = \left[ (H/L)\sqrt{1 - L^2/G^2} + (L/G)\sqrt{1 - H^2/G^2} \cos g \right] \cos l - \sqrt{1 - H^2/G^2} \sin g \sin l, \]
\[ \gamma_3 = HL/G^2 - \sqrt{1 - L^2/G^2} \sqrt{1 - H^2/G^2} \cos g. \] (2.11)

To construct the Poincaré section it was proposed to use a surface \(g = \text{const} [4–6]\). If the constant is zero, then, as seen from (2.10), in the original variables the section is defined by
\[ q = M_2\gamma_1 - M_1\gamma_2 = 0. \] (2.12)

3. METHODS OF NUMERICAL SOLUTIONS

The differential equations (2.1) are solved numerically by the 4th order Runge–Kutta finite-difference method [18]. At each integration step, to calculate the right-hand sides it is necessary to solve the linear algebraic equations 2.3 to evaluate the angular velocity components, and to use the relations (2.2) and (2.4) to compute all needed quantities.

The tests confirm that conservation of the integrals of motion (2.6) and (2.7) in the course of the numerical integration (with an appropriately chosen step) is provided with high precision. However, in some cases (for example, in calculations of the Lyapunov exponents, see below) it is useful to supply renormalization of the vector \(\gamma\) to get precisely the unit norm, and renormalization of the

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1) As the geometric integral is accounted in the definition, the total number of variables is reduced to five.
angular momentum $\mathbf{M}$ to get the prescribed fixed constant mechanical energy $E$. To do this, we substitute

$$
\gamma_i := \frac{\gamma_i}{\sqrt{\gamma_1^2 + \gamma_2^2 + \gamma_3^2}}, \quad k = i, 1, 2, 3, \tag{3.1}
$$

and

$$
M_i := M_i \sqrt{\frac{E + g_0(r_1\gamma_1 + r_2\gamma_2 + r_3\gamma_3)}{M_1\omega_1 + M_2\omega_2 + M_3\omega_3}}, \quad i = 1, 2, 3. \tag{3.2}
$$

(The latter procedure is omitted in those exceptional cases where the numerator in the expression under the square root turns out to be negative or close to zero due to the calculation errors.)

The Poincaré map calculation is organized as a special subroutine. To guarantee validity of the condition (2.12), the method of Hénon is applied at the end of the routine [13, 19]. Namely, if we get $q < 0$ at some step of the Runge–Kutta computations having $q > 0$ at the previous step, we provide an additional step using the same integration scheme, but taking $q$ as the independent variable and selecting the step size equal to $q$ with opposite sign. It ensures return of the system state just to the Poincaré section consistent in accuracy with the used finite-difference scheme.

To draw charts of dynamic regimes [13, 14] a procedure of scan of the parameter plane is carried out on a grid with some steps along two coordinate axes. At each point about $10^3$ iterations of the Poincaré map are produced, and the data for the final iteration steps are analyzed to detect the presence of some repetition period (from 1 to 14), up to some given small level of permissible error. If a certain period is detected, the pixel in the diagram is indicated by the corresponding color, and the routine proceeds with analysis of the next point in the parameter plane. To start iterations at the new point it is reasonable to use a state obtained as result of iterations at the previous point (“scan with inheritance”), with appropriate correction to get the prescribed energy $E$ if necessary. In most cases it helps to speed up essentially the convergence to sustained dynamics.

Let us discuss in detail the computation of spectrum of Lyapunov exponents [13, 20].

If we talk about the original system (2.1), it has six Lyapunov exponents, three of which are zero. One zero exponent is associated with infinitesimal shift along the reference trajectory, and the other two relate to infinitesimal shifts in the energy and in the norm of the vector $\gamma$. The remaining three non-trivial exponents are subject to computation, and we can do it using the Poincaré map.

For the Poincaré map, one zero exponent is immediately eliminated. Two others can be excluded too, if we apply the above-mentioned normalization of vector $\gamma$ to unit norm and of the momentum vector $\mathbf{M}$ to get the prescribed value of the total mechanical energy $E$.

A certain complexity of the original equations, which contain plurality of auxiliary variables, including the implicitly defined ones, forces us to prefer a method for computation of the Lyapunov exponents without explicit use of the linearized variational equations. Namely, for given parameters and initial conditions we perform simultaneously iterations of the Poincaré map (with renormalization of the vector $\gamma$ and the momentum vector $\mathbf{M}$) for four states with the same energy, one of which corresponds to the reference trajectory, and the other three to different weakly perturbed states. After each step of iteration, three perturbation vectors are orthogonalized with application of the Gram–Schmidt process, and normalized to some small prescribed fixed norm. Then, the computation is continued with the redefined three perturbed states. Three Lyapunov exponents are obtained as the rates characterizing the increase or decrease of cumulative sums of the logarithms of the ratios of the norms (after orthogonalization but before renormalization) to their initial values.

To determine the computation errors for the Lyapunov exponents, it is appropriate to use statistical analysis, such as that adopted in physics and other disciplines in the context of experimental evaluation of quantities in the presence of measurement errors [21]. To do this, the computations are executed repeatedly many times, say, on successive parts of the same phase trajectory, for equal numbers of iterations of the Poincaré map. The resulting set of data for each Lyapunov exponent is regarded as sample data for a random variable, for which the sample expectation and the sample variance may be calculated. The expectation is the statistical estimate of the Lyapunov exponent, and a reasonable measure of the calculation error is the standard deviation that is the square root of the sample variance (see examples in Section 6). Surely, one can apply more tools elaborated in the mathematical statistics (quantiles, confidence interval, Student
criterion), but it makes sense, apparently, only in the context of investigations specially focused on the analysis of statistical properties of the Lyapunov characteristics.

The presence of a positive Lyapunov exponent indicates a chaotic nature of the dynamics. If the sum of all three exponents is negative, the regime is interpreted as associated with a strange chaotic attractor. If the sum is about zero, up to calculation errors, we designate the dynamical behavior as the quasi-conservative chaos. The chaotic mixed dynamics mentioned in the introduction may be regarded as a particular case of the quasi-conservative chaos definitely associated with invariant set in the phase space symmetric with respect to the time reversal.

4. SOME NUMERICAL RESULTS: REPRODUCING THE PREVIOUS STUDIES

To ensure consistency with previous studies [5, 6, 9], we present here some illustrative results for the model of the rattleback (2.1). The principal radii of the curved surface of the rattleback are \( a_1 = 9 \), \( a_2 = 4 \). The distance from the top of the paraboloid to the center of mass of the body is \( h = 1 \), the gravitational acceleration is \( g_0 = 100 \), the principal moments of inertia are \( I_1 = 5 \), \( I_2 = 6 \), \( I_3 = 7 \), the angle between geometric and dynamical axes is \( \delta = 0.2 \).

Figure 2 illustrates one of the intriguing features of the rattleback — the reverse phenomenon. Here three components of the angular velocity are plotted versus time, as obtained in computations for the case of initial state corresponding to rotation around the vertical axis in the “wrong”

![Fig. 2. Dependence of the angular velocity versus time, illustrating the effect of reverse: after the transient, the direction of rotation, i.e. the component \( \omega_3 \) changes sign. The energy parameter \( E = 1380 \), the other parameters are \( I_1 = 5 \), \( I_2 = 6 \), \( I_3 = 7 \), \( g_0 = 100 \), \( a_1 = 9 \), \( a_2 = 4 \), \( h = 1 \), \( \delta = 0.2 \).]
direction with the energy $E = 1380$. It can be seen that small perturbations of the initial state give rise to development of a complex transient process, in which oscillations about other two axes appear, and then these vibrations transform the motion in such a way that the direction of rotation around the vertical axis is reversed; it corresponds to a change in sign of the component $\omega_3$.

Figure 3 shows portraits of dynamical regimes in the Andoyer–Deprit variables to demonstrate compliance with the numerical results of Ref. [9]. For each mode, the spectrum of Lyapunov

![Fig. 3](image_url)

Illustration of dynamic modes for the rattleback with parameters $I_1 = 5$, $I_2 = 6$, $I_3 = 7$, $g_0 = 100$, $a_1 = 9$, $a_2 = 4$, $h = 1$, $\delta = 0.2$ for various values of the energy:
(a) $E = 170$, a regime close to the situation of KAM tori, $\Lambda = (0.002, -0.000, -0.002)$
(b) $E = 220$, a “mixed mode”, or quasi-conservative chaos, $\Lambda = (0.028, -0.000, -0.028)$
(c) $E = 320$, a “mixed mode”, or quasi-conservative chaos, $\Lambda = (0.107, 0.000, -0.108)$
(d) $E = 470$, regime classified in Ref. [9] as a spiral attractor, $\Lambda = (0.184, -0.0002, -0.190)$
(e) $E = 510$, a stable cycle of period 5, $\Lambda = (-0.005, -0.015, -0.015)$, and its symmetric partner
(f) $E = 555$, quasi-conservative chaos $\Lambda = (0.109, -0.007, -0.123)$ and its symmetrical partner. The blue dots correspond to the attractors, and the red ones to the limit set corresponding to the dynamics in the reversed time, symmetric to the attractors with respect to the change of variables (2.8). All diagrams are presented in the Andoyer–Deprit variables for the Poincaré map, except the diagram (e), which shows the points at each integration step of numerical solution of the differential equations.
exponents is supplied in the caption, as obtained for the 3D Poincaré map. Attractors are shown in blue, and the red objects are repellers symmetric to them under the time-reversal operation (2.8).

Diagram (a) corresponds to $E = 170$, the regime similar to the situation of KAM tori (see [1]), where all three Lyapunov exponents are close to zero. Diagrams (b) and (c) correspond to “mixed mode” [9], or quasi-conservative chaos: the positive and negative Lyapunov exponents are practically the same in absolute value, and the remaining exponent is close to zero. Diagram (d) relates to the chaotic attractor of spiral type as classified in Ref. [9]; there is one positive Lyapunov exponent and two negative ones, and the sum of all the exponents is negative. Diagram (e) corresponds to a periodic regime, the cycle. In contrast to the other plots, here the phase trajectories in continuous time are shown. The blue curve corresponds to the attractor (the stable periodic orbit) with three negative Lyapunov exponents of the Poincaré map, and the red curve shows its symmetric partner, the repeller with three positive exponents. Finally, the diagram (f) relates to the quasi-conservative chaotic regime judging from the spectrum of Lyapunov exponents, but in this case a notable separation between the limit sets observed in forward and backward time is visible from the graph: they are not unified to a single, time-reversal symmetric object.

5. CHART OF DYNAMICAL REGIMES, BIFURCATION DIAGRAMS, LYAPUNOV EXPONENTS

For a more detailed discussion of the dynamics and the dependence on parameters, we select the case where the principal moments of inertia of the rattleback are $I_1 = 1$, $I_2 = 6$, $I_3 = 7$, and, according to Ref. [6], the picture of the dynamic behavior is quite rich.\footnote{This special choice of parameters corresponds to a particular situation where the “heavy” part of the rattleback has the form of a thin plate attached to a weightless body of convex shape that ensures the desired geometric properties. The “triangle inequality”, which must be valid for the principal inertia matrix components, in this case degenerates into equality.}

Figure 4 shows a chart of dynamical regimes for the Poincaré map of the system. The horizontal axis corresponds to the energy $E$, and the vertical axis to the angle $\delta$ specifying the angle of the principal geometric axis relative to the geometric axis. Drawing the chart we use data of the scan with inheritance from the left to the right and from the bottom to the top. The periods were determined from the dynamics of the component $M_3$ of the angular momentum. The legend for coding periods of motion by colors is shown in the right part of the figure.

Fig. 4. Chart of the dynamics of the Poincaré map for the rattleback model in the parameter plane. The horizontal axis corresponds to the energy $E$, and the vertical axis to the angle $\delta$. The other parameters are $I_1 = 1$, $I_2 = 6$, $I_3 = 7$, $g_0 = 100$, $a_1 = 9$, $a_2 = 4$, $h = 1$. Horizontal lines A and B correspond to the parameter variation in Fig. 5 and 6, respectively.
Black color means the absence of a certain identified period; the respective domains may correspond to the chaotic or quasi-periodic dynamics.

Areas in the left and top parts of the chart, shown in blue, i.e. identified as those of period 1 for $M_3$, correspond in fact to attractive orbits of period 2 symmetric with respect to the change of variables (2.9) for the complete state of the system $(M, \gamma)$. Thus, crossing the border between blue and green corresponds, in fact, not to the period doubling, but to the breaking symmetry bifurcation.

The distinguishing picture of edging of the black chaotic regions with narrow stripes (green – red – gray) corresponds to the sequences of period-doubling bifurcations accumulating at the border of chaos. Inside the black area, which occupies the central part of the chart, one can see a set of small-sized areas (“shrimps” in the terminology of Gallas [22]). The presence of these areas indicates the nonhyperbolic nature of chaos, similar to that in a wide class of dissipative systems, like the Hénon map and Ikeda map [13, 23].

In the middle of the bottom part of the chart, quasiperiodic regimes are observed with incommensurable ratios of involved oscillatory components together with interlaced mode locking regimes with rational frequency ratios. Here one can see a series of colored bands representing the mode locking periodicity (a kind of Arnold tongues). Examples of quasiperiodic regimes in this area are given below, in the discussion of Fig. 6.

Two horizontal lines A and B in Fig. 4 correspond to one-parameter bifurcation diagrams shown in Figs. 5 and 6 together with the plots of Lyapunov exponents versus parameter $E$.

The upper diagrams plot the $M_1$ component of the angular momentum at the crossings of the Poincaré section surface (2.10), versus the energy parameter $E$. The second picture is similar, but built for the component $M_3$. As the energy varies step by step, each time at the updated value of $E$ the initial conditions are redefined from the state at the previous energy by appropriate proportional conversion of three components of the angular momentum (see (3.2)). The data obtained by scanning with increase in energy and in backward direction are shown in different color (where the attractive sets appear to be different in both cases) that indicates the multistability, when two (or more) attractors coexist in the system. Each of the attractors has its own basin in the phase space, the set of initial states, starting from which the phase trajectories arrive at this attractor.

In Fig. 5 on the left side, and in Fig. 6 in the central part, periodic dynamics are observed, which correspond to a finite number of branches on the bifurcation tree, equal to the period of state repetition in the iterations of the Poincaré map. Here one can observe the multistability and the associated hysteresis effect: the final state of the system depends on the way the system arrives there, resulting in a distinction of the black and red lines. One can see the period-doubling bifurcations, where each branch undergoes a split, with emergence of periods 4 and 8. At such bifurcations the largest Lyapunov exponent vanishes, which also may be seen in the picture. Transition to chaos, at least on this diagram certainly is not of the type intrinsic to one-dimensional mappings [12, 13, 25]: instead of the gradual increase of the irregular component under parameter variation, intense chaos develops immediately (the possibility of such transition was noted also in [8]).

On the left side of the diagram in Fig. 6 chaotic dynamics takes place, as seen from the presence of the positive Lyapunov exponent. The second exponent is close to zero, and the third is negative and close in magnitude to the first one. Such a spectrum of Lyapunov exponents indicates the nature of the dynamic regime as the quasi-conservative chaos. It would not be correct to say that entirely in this region the dynamics correspond to objects in the state space invariant under time reversal; but we can say anyway that in the phase space the sets of orbits responsible for the asymptotic behavior in forward and backward time, are not significantly separated.

In the right part of the diagrams the chaotic regimes are not close to the quasi-conservative chaos, and have to be interpreted definitely as associated with strange attractors. It follows from the nature of the spectrum of Lyapunov exponents: there is one positive and two negative exponents for the Poincaré map, and the attractor has a well-defined Kaplan–Yorke dimension [12, 13, 24]. Certainly, it is not a hyperbolic attractor; one can observe a characteristic peculiarity in the behavior of parameter dependence for Lyapunov exponents, intrinsic to the non-hyperbolic chaos (especially well visible in Fig. 5). Namely, in the parameter range occupied mainly by chaos there occur deep narrow drops to negative values of the largest Lyapunov exponent, which are windows of periodicity
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Fig. 5. Bifurcation tree (a, b) and the graph of three Lyapunov exponents (c) for the three-dimensional Poincaré map of (2.1) with the parameters: $I_1 = 1$, $I_2 = 6$, $I_3 = 7$, $g_0 = 100$, $a_1 = 9$, $a_2 = 4$, $h = 1$, $\delta = \pi/4$. (or, windows of regularity) [12, 13, 22, 23]. In the chart of dynamical regimes it corresponds to crossing “shrimps” on the parameter plane.

In the presence of hysteresis and hard transitions, the chart of dynamical regimes can be imagined as a set of overlapping sheets, each of which corresponds to a particular attractor [13]. The sheets can be joined at some cusp points and have the edges at the fold lines running along the long tapered strips of periodic motions that extend far into the region occupied by chaos. These features are visible in Fig. 4, although the cusp points are not marked explicitly and can only be guessed. The above picture is qualitatively consistent with those observed in many dissipative systems capable of demonstrating complex dynamics and transitions to chaos.
Fig. 6. Bifurcation tree (a, b) and the graph of three Lyapunov exponents (c) for the three-dimensional Poincaré map of (2.1) with the parameters: $I_1 = 1, I_2 = 6, I_3 = 7, g_0 = 100, \alpha_1 = 9, \alpha_2 = 4, h = 1, \delta = 3\pi/16$.

Figure 7 shows again the same chart of dynamic regimes, but here the phase portraits for the Poincaré map are shown around the picture to illustrate dynamics at representative points in the parameter plane. They are depicted in the Andoyer-Deprit variables (see notation of the coordinate axes in the diagram (a)). In each case the attractor is shown in blue. Red corresponds to repellers, the limit objects that appear in the iterations in reverse time, which are symmetric partners of the “blue” objects relative to the operation of time reversal (8).

Diagram (a) corresponds to a stable cycle of period 2; the symmetric partner is an unstable cycle of the same period.

Diagram (b) represents a strange attractor and symmetric repeller.

Diagram (c) corresponds to one of the periodicity windows in the region occupied mostly by chaos. The period of the stable cycle is 14; there is also a symmetric partner, the unstable cycle of the same period.
Fig. 7. Chart of dynamical regimes (center) in the parameter plane for the rattleback with $I_1 = 1$, $I_2 = 6$, $I_3 = 7$, $g_0 = 100$, $a_1 = 9$, $a_2 = 4$, $h = 1$ and phase portraits of sustained dynamics at representative points for the Poincaré map in the Andoyer–Deprit variables. Each portrait has a subscript indicating the spectrum of Lyapunov exponents of the attractor of the Poincaré map for this mode.

Diagram (d) refers to a cycle of period 4. There is also a symmetric partner, unstable cycle of the same period.

Diagrams (e) and (f) correspond to the presence of a strange attractor and a symmetric repeller.

Diagram (g) represents the object that appears to be symmetric with respect to time reversal. Judging from the spectrum of Lyapunov exponents, containing three zeros (up to calculation errors) it is a quasi-periodic regime. In the phase space of the continuous time system it corresponds
to three-frequency quasiperiodicity associated with an attractive three-dimensional torus, which belongs entirely to the hypersurface of constant integrals of motion.

In diagram (h) one can see an invariant closed attractive curve and, as a symmetrical partner, the repellent invariant curve. This corresponds to a two-frequency quasiperiodic motion and to an attractive two-dimensional torus on the hypersurface of constant integrals of motion in the phase space of the continuous time system.

Diagrams (i) and (k) correspond to chaotic regimes, in which the positive Lyapunov exponent and the negative exponent are almost the same in magnitude, and the remaining one is close to zero. Hence, we regard them as quasi-conservative chaos. Closer inspection of the pictures shows that one cannot be sure about the symmetry of the limit set under time reversal; the figures contain many entangled but distinguishable sets formed by blue or red dots. Since the blue and red points are situated more or less in the same regions of the phase space, the Lyapunov exponents obtained from averaging over characteristic time period are close for the limit motions in forward and reverse time.

6. STRANGE ATTRACTORS AND THE TRANSITION TO CHAOS

In this section we discuss in more detail some strange attractors in the nonholonomic model of the rattleback. Here the principal moments of inertia are assigned as $I_1 = 2$, $I_2 = 6$, $I_3 = 7$. Parameters $E$ and $\delta$ are varied, and the other parameters remain the same as those in the previous sections.

Figure 8 shows a chart of dynamical regimes in the plane of the energy $E$ and the angle parameter $\delta$. The legend is the same as that in Figure 4. The attractors, which will be specially discussed, correspond to three points in the parameter plane:

A: $E = 642$, $\delta = 0.922$,
B: $E = 770$, $\delta = 0.405$,
C: $E = 620$, $\delta = 3\pi/8 = 1.178\ldots$

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig8.png}
\caption{Parameter plane chart of the Poincaré map for the rattleback model. The horizontal axis corresponds to the energy $E$, and the vertical axis to the angle parameter $\delta$. The other parameters are $I_1 = 2$, $I_2 = 6$, $I_3 = 7$, $g_0 = 100$, $a_1 = 9$, $a_2 = 4$, $h = 1$.}
\end{figure}

If we move up in the parameter plane along a vertical line passing through the point A, transition to chaos is observed through a sequence of period doubling bifurcations. It is illustrated in the bifurcation tree diagram in Fig. 9, where one can see the characteristic disposition of branches undergoing splitting at the bifurcation points, and “crown” filled with dots corresponding to chaos. The insets show fragments of the picture with magnification. In larger scale the tree looks more
and more similar to the classic structure of the “Feigenbaum tree”, well known for one-dimensional maps and for other highly dissipative systems with period doubling transition to chaos [12, 13].

By measuring the distances between the consecutive bifurcations (one can do this directly in pixels using diagrams depicted in appropriate scale), and calculating their ratios for successive levels of doubling, we obtain the values \( \delta_F \) enlisted in the first row of Table 1. Similarly, by measuring the ratios of the vertical splitting of the branches, we fill in the second row of the table. If the transition belongs to the Feigenbaum universality class [13, 25, 26], the values of the ratios should converge to the scaling constants \( \delta_F = 4.669201 \ldots \) and \( \alpha_F = -2.502907 \ldots \). As seen from the Table, this assumption agrees well with the data.

<table>
<thead>
<tr>
<th></th>
<th>(2,4)/(4,8)</th>
<th>(4,8)/(8,16)</th>
<th>(8,16)/(16,32)</th>
<th>(32,64)/(64,128)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta_F )</td>
<td>6.52</td>
<td>5.29</td>
<td>4.70</td>
<td>4.62</td>
</tr>
<tr>
<td>( \alpha_F )</td>
<td>-3.79</td>
<td>-2.82</td>
<td>-2.64</td>
<td>-2.54</td>
</tr>
<tr>
<td>Feigenbaum constants</td>
<td>4.669201</td>
<td></td>
<td></td>
<td>-2.502908</td>
</tr>
</tbody>
</table>

Observe that the ratios obtained at the initial levels of doubling are larger in absolute values than the expected \( \delta_F \) and \( \alpha_F \). It happens due to the so-called crossover phenomenon [27]. At relatively low effective dissipations (in our case it is understood as characteristic compression for 3D phase space volume in the course of dynamic evolution in the respective region), the ratios are close to those intrinsic to period doubling in conservative systems (\( \delta_H = 8.7211 \ldots \) and \( \alpha_H = -4.018 \ldots \) [26, 28]). At each successive period doubling, the effective dissipation is doubled, so the values of the ratios approach the Feigenbaum universal constants \( \delta_F \) and \( \alpha_F \).

Since the analysis indicates that the transition to chaos observed here relates to the Feigenbaum universality class, we can assume that at large characteristic times the dynamics can be reduced, in fact, to some unimodal one-dimensional map with a quadratic extremum (because of the rather strong contraction in the phase space of the three-dimensional map in the domain, where the attractor lies). Consequently, in a neighborhood of the critical point all phenomena should occur, which are characteristic for one-dimensional maps. In particular, in accordance with the result of Jacobson [29], in the supercritical parameter range there must be a set of positive measure where the system exhibits chaotic dynamics. For the original 3D map it corresponds to strange attractors similar to those in the Hénon map with strong dissipation, according to the results of Benedicks and Carleson [30].
The chaotic attractor resulting from the cascade of period-doubling bifurcations is shown in Fig. 10 that corresponds to the point A \((E = 642, \delta = 0.922)\). This attractor of the three-dimensional Poincaré map is shown in projections on the planes \((M_1, M_2)\) and \((M_1, M_3)\). Visually, it looks like attractors of dissipative maps, observed immediately after the transition to chaos through the period doubling cascade \([13]\).

Spectrum of Lyapunov exponents for the Poincaré map of this attractor is

\[
\Lambda_1 = 0.0453 \pm 0.002, \quad \Lambda_2 = -0.0845 \pm 0.0003, \quad \Lambda_3 = -0.3112 \pm 0.002,
\]

where the standard deviations are indicated as errors obtained for a number of samples of the data.\(^3\) As one can see, there is one positive exponent and two negative ones, which exceed the first one in absolute value. Then, the estimate of the dimension via the Kaplan–Yorke formula \([12, 13, 24]\) requires taking into account of only two larger exponents:

\[
D = m + \left( \sum_{i=1}^{m+1} \Lambda_i \right) / |\Lambda_{m+1}| = 1 + \Lambda_1 / |\Lambda_2| \approx 1.53.
\]

Figure 11 shows the power spectrum for the variable \(M_3\) generated by the dynamics on the attractor. The spectrum was obtained using the standard procedure \([31]\) by processing time series generated by the system (2.1), with sampling interval containing few integration steps of integration of the differential equations. The spectrum looks typical for attractors emerging through the Feigenbaum period doubling cascade. Indeed, there is a set of peaks with a hierarchical structure. The peaks of each next level of the hierarchy should be about 13.4 dB lower than the previous one \([12, 13]\). It is in reasonable agreement with the observed pattern. On deeper levels peaks are destroyed, and a continuous spectrum appears there according to the fact that the dynamics is chaotic.

\(^3\)The calculations were performed for 15 samples of data corresponding to successive fragments of the same trajectory on the attractor, each containing \(5 \cdot 10^3\) iterations of the Poincaré map; after that the expected values and standard deviations are calculated for the estimated values.
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Figure 11. The Fourier spectrum of the component $M_3$ on the attractor for the parameters $I_1 = 2$, $I_2 = 6$, $I_3 = 7$, $g_0 = 100$, $a_1 = 9$, $a_2 = 4$, $h = 1$, $E = 642$, $\delta = 0.922$. A logarithmic scale is used.

Figure 12 shows the chaotic attractor at the point B ($E = 770$, $\delta = 0.405$). The visual inspection suggests the origin of this attractor from destruction of an invariant torus (the last would correspond to a closed invariant curve in the Poincaré section). In Figure 13 the Fourier spectrum is plotted for the signal generated by the dynamics of the component $M_3$ on this attractor. The spectrum is very uneven, but definitely contains notable continuous component corresponding to chaotic dynamics. The Lyapunov exponents for this attractor of the 3D map are

$$\Lambda_1 = 0.077 \pm 0.004, \quad \Lambda_2 = -0.100 \pm 0.003, \quad \Lambda_3 = -0.128 \pm 0.002,$$

and its dimension estimate by the Kaplan–Yorke formula is

$$D = 1 + \Lambda_1 / |\Lambda_2| \approx 1.76.$$  

Figure 14 shows a chaotic attractor at the point C ($E = 620$, $\delta = 3\pi/8$). The Lyapunov exponents are

$$\Lambda_1 = 0.282 \pm 0.009, \quad \Lambda_2 = -0.093 \pm 0.004, \quad \Lambda_3 = -0.686 \pm 0.007.$$  

The first Lyapunov exponent is positive, and the others are negative. The absolute value of the second exponent is less than that of the first one. Therefore, the dimension evaluated from Kaplan–Yorke formula is greater than 2:

$$D = 2 + (\Lambda_1 + \Lambda_2) / |\Lambda_3| \approx 2.26.$$  

So, this is an essentially higher-dimensional attractor than those discussed above; moreover, visually it looks essentially different from the portraits in Figs. 10 and 12. Judging from the available
numerical results, attractors of this type are observed sufficiently often in the nonholonomic model of the rattleback (in contrast to many dissipative chaotic systems, where the first negative Lyapunov exponent is usually distant enough from zero.) Figure 15 shows the Fourier spectrum of the signal generated by the dynamics of the component $M_3$ on this attractor. The spectrum is certainly continuous and corresponds to chaotic dynamics. Unlike the spectra in Figs. 11 and 13 it does not contain notable peaks; it indicates the absence of any significant periodic components of the motion.

Fig. 13. The Fourier spectrum of the component $M_3$ on the attractor for the parameters $I_1 = 2$, $I_2 = 6$, $I_3 = 7$, $g_0 = 100$, $a_1 = 9$, $a_2 = 4$, $h = 1$, $E = 770$, $\delta = 0.405$. A logarithmic scale is used.

Fig. 14. Portrait of the attractor in the Poincaré section for the parameters $I_1 = 2$, $I_2 = 6$, $I_3 = 7$, $g_0 = 100$, $a_1 = 9$, $a_2 = 4$, $h = 1$, $E = 620$, $\delta = 3\pi/8$, in two projections for the angular momentum.

Fig. 15. The Fourier spectrum of the component $M_3$ on the attractor for the parameters $I_1 = 2$, $I_2 = 6$, $I_3 = 7$, $g_0 = 100$, $a_1 = 9$, $a_2 = 4$, $h = 1$, $E = 620$, $\delta = 1.178$. A logarithmic scale is used.
7. DIMENSIONS OF THE ATTRACTORS

A common method of quantitative characterization of strange attractors is to estimate dimensions:

- the fractal dimension, called also the capacity, or the box counting dimension,
- the information dimension,
- the correlation dimension,

by means of processing time series obtained in the course of operation of the system [12, 13, 32, 33]. It is interesting to bring this approach to the attractors associated with the nonholonomic model of the rattleback.

To estimate the capacity dimension, the phase space domain containing the attractor is partitioned into boxes of equal size, and the number of cells covering the attractor is considered in dependence on the size of the cells. To estimate the information dimension, one evaluates probabilities of residence of a typical trajectory in the attractor in the cells and analyzes the dependence of amount of information concerning visiting the cells on the size of the cells. Finally, to evaluate the correlation dimension, one considers probabilities for repeated visits of the partition cells depending on the size of the cells.

In our computations a segment of trajectory was generated by $M = 10^6$ iterations of the Poincaré map for the nonholonomic rattleback model, and the data in the Andoyer–Deprit variables $(H/G, L/G, l)$ were stored. The 3D space $(H/G, L/G, l)$ was covered by cubic cells of edge length $\varepsilon_k = 1/2^k$ along the coordinate axes; at the $k$th level the total number of the cells was $2^{3k}$ ($k = 1, \ldots, 13$). For ith cell the probability of residence is estimated as $p_i = M_i / M$, where $M_i$ is a number of hits in this cell for the orbit we deal with.

Let $N(\varepsilon)$ be a number of cells of size $\varepsilon$ covering the attractor. The capacitive, the information and the correlation dimensions are defined as

$$D_0 = -\lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log \varepsilon} \quad \text{(capacitive)}, \quad (7.1)$$

$$D_1 = -\lim_{\varepsilon \to 0} \frac{I(\varepsilon)}{\log \varepsilon}, \quad \text{where} \quad I(\varepsilon) = -\sum_{i=1}^{N(\varepsilon)} p_i \log p_i \quad \text{(information)}, \quad (7.2)$$

$$D_2 = \lim_{\varepsilon \to 0} \frac{\log C(\varepsilon)}{\log \varepsilon}, \quad \text{where} \quad C(\varepsilon) = \sum_{i=1}^{N(\varepsilon)} p_i^2 \quad \text{(correlation)}. \quad (7.3)$$

For numerical evaluation of these dimensions, we plot $N(\varepsilon_k)$, $I(\varepsilon_k)$, and $C(\varepsilon_k)$ versus $\varepsilon_k$ in the double logarithmic scale. In each graph one can single out a range of linear behavior. The slope of the straight line approximating this part of the dependence estimates the respective dimension. Results for attractors discussed in the previous section (relating to the points B and C in the parameter space) are shown in Fig. 16.

In both cases, the dimensions are fractional; it indicates the fractal nature of the attractors. For the attractor at the point B we have the information dimension 1.60, and the Kaplan–Yorke dimension is 1.76. For the attractor at the point C they are, respectively, 2.26 and 2.0. It may be considered as quite reasonable agreement.

8. CONCLUSION

In this paper we have presented results of numerical study of the rattleback dynamics exploiting a variety of methods used previously for analysis of complex dynamics and chaos in dissipative systems, which are adapted here for the nonholonomic model.

First, we have demonstrated compliance of our techniques with the previously known numeric results.
Fig. 16. Results of evaluation of capacitive (left), information (in the center) and correlation (right) dimensions for attractors of the 3D Poincaré map of the nonholonomic model of the rattleback at $E = 770$, $\delta = 0$ (a) and $E = 620$, $\delta = 3\pi/8$ (b). The other parameters are $I_1 = 2$, $I_2 = 6$, $I_3 = 7$, $g_0 = 100$, $a_1 = 9$, $a_2 = 4$, $h = 1$.

Next, we have obtained and interpreted charts of dynamical regimes in the parameter plane of the rattleback. Some characteristic features of the parameter plane structures, previously observed only for dissipative systems, are revealed here, including characteristic shapes of the domains of period doubling near the chaos border and of islands of periodicity within the area occupied by chaos (“shrimp”). Also we note the presence of quasi-periodic regimes associated with attractive two-dimensional and three-dimensional tori in the phase space of the system reduced to the 3D mapping. We have demonstrated efficiency of analysis based on drawing one-parameter bifurcation diagrams (“bifurcation trees”), in particular, it easily reveals domains of multistability, i.e. of coexistence of two or more attractors for the same values of the parameters of the system.

We have developed and implemented a method for calculating the full spectrum of Lyapunov exponents taking into account the features of the nonholonomic model. The analysis of the Lyapunov exponents of chaotic regimes reveals two classes of chaotic motions, one corresponding to strange attractors (one positive and two negative Lyapunov exponents), and the other to quasi-conservative dynamics (close magnitude of the positive and the negative exponents, while the remaining one is close to zero). Calculations show that the first type of behavior is typical for regions of relatively large, and the second for regions of relatively small values of energy. A particular case of chaos of the second kind is the so-called mixed dynamics, which corresponds in the phase space to a set of trajectories invariant under time reversal.

Several representatives of strange attractors in the nonholonomic model of the rattleback are studied in detail; we have presented their portraits in the Poincaré section, Lyapunov exponents, Fourier spectra, and dimension estimates. It is demonstrated that the occurrence of one of these attractors arising through the period doubling bifurcation cascade is associated with the Feigenbaum class of quantitative universality, the same as that for 1D unimodal quadratic maps.
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