Weak Limits of Probability Distributions in Systems with Nonstationary Perturbations

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Consider a system of differential equations

\[
\dot{x} = \omega, \quad \dot{\omega} = f(t),
\]

where \(x = (x_1, x_2, \ldots, x_n, \text{mod } 2\pi)\) are angular coordinates on an \(n\)-dimensional torus, \(\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in \mathbb{R}^n\), and \(f\) is a given vector function of \(t\). Assume that \(f\) is twice (Riemann) integrable with respect to time \(t\). Equations (1) describe the motion of a classical system with configuration space \(\mathbb{T}^n = \{x\}\) and kinetic energy \(T = \frac{(\omega_1, \omega_2, \ldots, \omega_n)}{2}\) under the action of an external force \(f\).

If \(f = 0\), then (1) is a completely integrable Hamiltonian system, with the coordinates \(x\) and \(\omega\) being action-angle variables. The same form is possessed by perturbations of completely integrable Hamiltonian systems in the general nondegenerate case.

Following Gibbs, we define a probability measure \(\rho(x, \omega) d^n x d^n \omega\) with a summable density \(\rho\) in the phase space \(\Gamma = \mathbb{T}^n \times \mathbb{R}^n\). The flow of system (1) transports this measure, so that the density \(\rho(x, \omega)\) becomes a function of time. Since the divergence of the right-hand side of system (1) is zero, the probability density satisfies the Liouville equation

\[
\frac{\partial \rho}{\partial t} + \left( \frac{\partial \rho}{\partial x}, \omega \right) + \left( \frac{\partial \rho}{\partial \omega}, f \right) = 0 \tag{2}
\]

with initial condition \(\rho_0 = \rho\).

Let \(\varphi : \mathbb{T}^n \to \mathbb{R}\) be a measurable bounded function. Since \(\rho_t \in L_1(\Gamma)\) for all \(t\), the integral

\[
K(t) = \int_{\mathbb{T}^n} \rho_t(x, \omega) \varphi(x) d^n x d^n \omega
\]

is a well-defined function of time. If \(\varphi\) is the characteristic function of a measurable domain \(D \subset \mathbb{T}^n\), then \(K(t)\) is the fraction of Hamiltonian systems in the Gibbs ensemble that occupy \(D\) at time \(t\).

According to the ergodic theorem, the limit

\[
\lim_{t \to \pm \infty} \frac{1}{\tau} \int_0^\tau \rho(x - \omega t, \omega) dt \tag{3}
\]

exists for almost all \(x\) and \(\omega\), coincides almost everywhere with an integrable function \(\bar{\rho}(\omega) \geq 0\), and

\[
\int_{\mathbb{R}^n} \bar{\rho} d^n x d^n \omega = (2\pi)^n \int_{\mathbb{R}^n} \bar{\rho}(\omega) d^n \omega = 1. \tag{4}
\]

Thus, the function \(\bar{\rho}\) can be treated as the density of the limit probability measure (in a weak sense) that corresponds to a statistical equilibrium of the system under consideration.

THE MAIN RESULT

Theorem 1. Under the assumptions made above,

\[
\lim_{t \to \pm \infty} K(t) = \int_{\mathbb{T}^n} \bar{\rho}(\omega) \varphi(x) d^n x d^n \omega
\]

\[
= \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \varphi(x) d^n x. \tag{5}
\]

Corollary. Let \(\varphi\) be the characteristic function of a measurable domain \(D\). Then

\[
\lim_{t \to \pm \infty} K(t) = \frac{\operatorname{mes} D}{\operatorname{mes} \mathbb{T}^n}.
\]

Thus, as time increases indefinitely, the systems in the Gibbs ensemble become uniformly distributed on the \(n\)-dimensional configuration torus \(\mathbb{T}^n\). For \(f = 0\), this result was established in [1].

Theorem 1 is proved by the method described in [1]. The basic point lies in the analysis of the case where \(\varphi(x) = \exp(i(m, x))\), \(m \in \mathbb{Z}^n\). It is necessary to show that, for \(m \neq 0\),

\[
\int_{\mathbb{T}^n} \rho_t(x, \omega) e^{i(m, x)} d^n x d^n \omega \to 0 \tag{5}
\]

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as \( t \to \pm \infty \). For this purpose, we first solve the Liouville equation (2):

\[
\rho_{i}(x, \omega) = \rho(x - \omega t + h(t), \omega - g(t)),
\]

where \( \rho \) is a Cauchy datum, \( g(t) = f(t), \; g(0) = 0, \)
\( h(t) = tf(t), \) and \( h(0) = 0. \) Formula (6) is verified by
direct calculations.

Thus,

\[
K(t) = \int_{\Gamma} \rho(x - \omega t + h, \omega - g) \phi(x) d^n x d^\omega
\]

\[
= \int_{\Gamma} \rho(x, \omega) \phi(x + \omega t + \lambda(t)) d^n x d^\omega,
\]

where \( \lambda(t) = g(t) \) and \( \lambda(0) = 0. \) It is easy to verify that
\( \dot{\lambda} = -h. \)

Now setting \( \phi = \exp(i m, x) \), we derive an explicit
formula for the integral in (5):

\[
e^{i(m, \lambda)} \int_{\mathbb{R}^*} \rho(x, \omega) e^{i(m, \phi(x))} d^n x d^\omega
\]

\[
= e^{i(m, \lambda)} \int_{\mathbb{R}^*} \rho_{m}(\omega) e^{i(m, \phi(x))} d^\omega,
\]

(7)

where

\[
\rho_{m}(\omega) = \int_{\mathbb{R}^*} \rho(x, \omega) e^{i(m, x)} d^\omega.
\]

Since \( \rho_{m} \) is an integrable function, we conclude that, for
\( m \neq 0 \), integral (7) approaches zero as \( t \to \pm \infty \) (according
to the theory of the Fourier transform), which was
to be proved.

Remark. In the presence of a force \( f \), an additional
bounded oscillating factor \( \exp(i m, x) \) appears in (7).

Theorem 1 can be extended in different directions.
For example, suppose that the initial density \( \rho \) belongs
to \( L_2(\Gamma) \) (hence, \( \rho \in L_2 \) for all \( t \)) and \( \phi \) is a function
from \( L_2(\Gamma) \). Then

\[
K(t) = \int_{\Gamma} \rho \phi d^n x d^\omega
\]

(8)

is a well-defined function of time. It happens that

\[
\lim_{t \to \pm \infty} K(t) = \int_{\Gamma} \bar{\rho} \phi d^n x d^\omega,
\]

(9)

where \( \bar{\rho} \) is defined by limit (3). Thus, \( \bar{\rho} \) is a weak limit
of \( \rho \) as time increases indefinitely. The state of the system
with probability density \( \bar{\rho} \) can be called a statistical
(thermal) equilibrium. It should be emphasized that
the presence of a nonstationary perturbing force \( f(t) \)
does not influence the approach of the system to ther-
al equilibrium.

Let

\[
S(t) = -\int_{\Gamma} \rho \ln \rho_d^n x d^\omega
\]

be the entropy of the system at time \( t \). It is easy to show
that \( S(t) \equiv \text{const}. \) This is a generalization of Poincaré’s
observation that the fine-grained entropy of autonomous
dynamic systems is constant (see [2]). It is possible
to introduce the entropy of a system at statistical
equilibrium:

\[
S_{\infty} = -\int_{\Gamma} \bar{\rho} \ln \bar{\rho} d^n x d^\omega.
\]

We have the simple inequality

\[
S(t) \leq S_{\infty},
\]

which corresponds to the second law of thermodynamics
for irreversible processes. The formula for the
entropy increment \( S_{\infty} - S_{\infty} \) can be derived in accordance
with phenomenological thermodynamics (a dis-
cussion can be found in [1]). However, in the general
case, inequality (10) is valid only for adiabatic pro-
cesses, without any heat inflow. For the system con-
sidered, \( T = (\omega, f) \neq 0. \)

Note that the integral in (8) is also defined when \( \rho \in
L_p(\Gamma) \) and \( \phi \in L_q(\Gamma) \), where
\( \frac{1}{p} + \frac{1}{q} = 1 \). The limit rela-
tion (9) is also true in this case. In Theorem 1, \( p = 1 \) and
\( q = \infty \) (recall that \( L_\infty \) is the class of essentially bounded
measurable functions).

SINGULAR LIMIT DISTRIBUTIONS

Consider the simple problem of oscillations of a unit-mass ball between two walls \( 0 \leq z \leq a. \) Suppose
that a force \( f(t) \) acts on the ball. For example, we may
assume that a charged ball is placed in a variable elec-
tric field. At first glance, this is a system of type (1)—
an external perturbation of an integrable system. How-
ever, this is not the case, and the problem is reduced to
the analysis of parametric perturbations.

Consider a two-sheeted cover of the line segment by
the circle \( \mathbb{T} = \{ x \mod 2\pi \} \), introducing an angular vari-
iable according to the following rule: \( x = \frac{\pi z}{a} \) when \( z
\]

increases from zero to \( a, \) and \( x = 2\pi - \frac{\pi z}{a} \) when \( z
\]
designs from \( a \) to zero. The equation of motion of the
ball takes the form

\[
\dot{x} = -f(t) V_x,
\]

(11)

where \( V(x) = -\frac{\pi x}{a} \) for \( 0 < x < \pi \) and \( V(x) = \frac{\pi x}{a} - \frac{2\pi^2}{a} \)
for \( \pi < x < 2\pi. \) The evolution of probabilities of the
measure of Eq. (11) is a more complicated problem.
[compared to the analysis of system (1)], and it can be solved only under some additional conditions.

For example, let \( f(t) = \text{const.} \) Then Eq. (11) can be explicitly integrated, and it is easy to show that the weak limit of the probability density of the measure is a function of the total energy \( \frac{x^2}{2} + fV(x) \). Integration with respect to velocity yields a probability density in the configuration space, which is generally not constant (see [1]).

Assume that \( f(t) \) increases monotonically as \( t \to +\infty \) and

\[
\frac{df}{dt} \leq \frac{3}{2} f^2. \tag{12}
\]

Applying the method of [3], we can show that all solutions \( x(t) \) to Eq. (11) tend to the minimum point of the potential \( V(x) \) as \( t \to +\infty \). Consequently, under these assumptions, the limit probability density of the ball’s positions on the line segment coincides with the delta function \( \delta(z - a) \).

These observations can be generalized. Suppose that \( M^n = \{x\} \) is the compact configuration space of a mechanical system with \( n \) degrees of freedom, \( T \) is the kinetic energy [a positive definite quadratic form in the momenta \( y = (y_1, y_2, \ldots, y_n) \)], \( V: M \to \mathbb{R} \) is a smooth function, and \( f(t)V \) is the potential energy. The phase space \( \Gamma \) is the cotangent bundle of \( M \), and the Hamiltonian is \( H = T + f(t)V \). Let \( \rho_\tau \) be the probability density in \( \Gamma \) transported by the flow of the Hamiltonian system, and let \( \rho_0 = \rho \) be a Cauchy datum.

**Theorem 2.** Suppose that the measure \( \rho d^nxdn\omega \) is absolutely continuous with respect to the Liouville measure on \( \Gamma \), the function \( V \) has only nondegenerate critical points on \( M \), the function \( t \mapsto f(t) \) increases monotonically with \( t \), and (12) is fulfilled. If \( \phi: M \to \mathbb{R} \) is the characteristic function of a measurable domain on \( M \) not containing local minimum points of \( V \), then

\[
\int_{\Gamma} \rho_\tau(x, y) \phi(x) d^nxdn\omega \to 0
\]

as \( t \to +\infty \).

**CONCLUSIONS**

Thus, the limit distribution of the Gibbs ensemble on the configuration space \( M \) is singular: this measure is concentrated on a finite set of points that are local minima of \( V \). Theorem 2 is deduced from the result of [3]: under the conditions specified, almost all solutions to the Hamilton equations with the Hamiltonian \( H = T + fV \) are such that \( x(t) \) tends to a local minimum of \( V \) as time increases indefinitely. Moreover, the momenta \( y(t) \) are unbounded (by the Liouville theorem on the conservation of the phase volume of Hamiltonian systems). Therefore, the frequencies of small-amplitude oscillations increase indefinitely as the system approaches a stable equilibrium.

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**REFERENCES**