Absolute and Relative Choreographies in the Problem of the Motion of Point Vortices in a Plane

A. V. Borisov, A. A. Kilin, and I. S. Mamaev

Received September 2, 2004

Presented by Academician V.V. Kozlov June 9, 2004

Institute of Computer Science, Universitetskaya ul. 1, Izhevsk, 426034 Russia
e-mail: borisov@rcd.ru, aka@rcd.ru, mamaev@rcd.ru

MATHEMATICAL PHYSICS

EQUATIONS OF MOTION AND FIRST INTEGRALS

We briefly outline the basic forms of the equations and first integrals for the dynamics of point vortices moving in a plane (a more detailed explanation can be found [1–3], where the fluid dynamic assumptions under which these equations are valid are also stated).

The equations of motion for \( n \) point vortices with Cartesian coordinates \((x_i, y_i)\) and intensities \( \Gamma_i \) can be written in Hamiltonian form as

\[
\Gamma_i \dot{x}_i = \frac{\partial \mathcal{H}}{\partial y_i}, \quad \Gamma_i \dot{y}_i = -\frac{\partial \mathcal{H}}{\partial x_i}, \quad 1 \leq i \leq n, \tag{1}
\]

with the Hamiltonian

\[
\mathcal{H} = -\frac{1}{4\pi} \sum_{i<j}^n \Gamma_i \Gamma_j \ln |\mathbf{r}_i - \mathbf{r}_j|^2, \quad \mathbf{r}_i = (x_i, y_i). \tag{2}
\]

Here, the Poisson bracket is given by

\[
\{ f, g \} = \sum_{i=1}^n \frac{1}{\Gamma_i} \left( \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial x_i} \right). \tag{3}
\]

Equations (1) are invariant under the group of motions of the plane \( E(2) \). Therefore, in addition to the Hamiltonian, they have the three integrals

\[
Q = \sum_{i=1}^n \Gamma_i x_i, \quad P = \sum_{i=1}^n \Gamma_i y_i, \quad I = \sum_{i=1}^n (x_i^2 + y_i^2), \tag{4}
\]

which are, however, not involutive:

\[
\{ Q, P \} = \sum_{i=1}^n \Gamma_i, \quad \{ P, I \} = -2Q, \quad \{ Q, I \} = 2P. \tag{5}
\]

From the three integrals, we can form two involutive ones: \( Q^2 + P^2 \) and \( I \). Therefore, according to the general theory [4], the number of degrees of freedom of the system can be reduced by two degrees of freedom. Specifically, the three-vortex system is reduced to a system with one degree of freedom and is integrable [Gröbli, Kirchhoff, Poincaré] [1–3], and the four-vortex system is reduced to a system with two degrees of freedom. In general, the latter problem is not integrable [5].

In the case of equal intensities, the most convenient (for our purpose) reduction scheme for \( n \) vortices in a plane was suggested in [6] (see, also, [1, 7]). The scheme is based on representing the equations of motion for point vortices in a plane in terms of mutual variables (going back to E. Laura), namely, the squared distances between the pairs of vortices and oriented areas of the vortex triangles:

\[
M_{ij} = (x_i - x_j)^2 + (y_i - y_j)^2, \quad \Delta_{ijk} = (\mathbf{r}_j - \mathbf{r}_k) \wedge (\mathbf{r}_k - \mathbf{r}_i). \tag{6}
\]

The mutual commutation of these variables gives a Lie algebra. The order-reduction procedure (more exactly, the last “canonical” step in this procedure) is reduced to the purely algebraic problem of introducing symplectic coordinates on orbits of the corresponding Lie algebras.

ORDER REDUCTION FOR THREE AND FOUR VORTICES OF EQUAL INTENSITY

Without loss of generality, we set \( \Gamma_i = \Gamma_i = 1 \) and \( P = Q = 0 \). Then, the integral of moment \( I \) in (4) can be represented as

\[
I = \frac{1}{n} \sum_{i<j} M_{ij}, \tag{7}
\]

where \( n \) is the number of vortices. Representing the coordinates of the vortices in the complex form \( z_k = x_k + iy_k \), we obtain the representation

\[
z_k = \frac{1}{n} \sum_{j \neq k} \sqrt{M_{kj} e^{i\theta_{kj}}}, \tag{8}
\]

where \( \theta_{kj} \) is the angle between the vector directed from the \( j \)th to \( k \)th vortex and the positive \( Ox \) axis.
Direct calculations prove the following statements describing the dynamics of the reduced system of three or four vortices.

**Proposition 1.** For three vortices of equal intensity, the evolution of their relative distances (assuming that \( I = \text{const} \)) is described by a Hamiltonian system with one degree of freedom. In canonical variables \((g, G)\), that system is written as

\[
g = \frac{\partial \mathcal{H}}{\partial G}, \quad \dot{G} = -\frac{\partial \mathcal{H}}{\partial g}, \quad \mathcal{H} = -\frac{1}{4\pi} \ln M_{12} M_{13} M_{23}, \quad (9)
\]

where

\[
M_{12} = 4\left(\frac{1}{2} - G\right),
\]

\[
M_{13} = 8G - I + 2\sqrt{12} \left(\frac{1}{2} - G\right)G \cos g,
\]

\[
M_{23} = 4\left(\frac{1}{2} - G\right) - 2\sqrt{12} \left(\frac{1}{2} - G\right)G \cos g.
\]

**Proposition 2.** For four vortices of equal intensity, the evolution of their relative distances is described by a Hamiltonian system with two degrees of freedom. In canonical variables \((g, G, h, H)\), that system is written as

\[
g = \frac{\partial \mathcal{H}}{\partial G}, \quad \dot{G} = -\frac{\partial \mathcal{H}}{\partial g}, \quad \dot{h} = \frac{\partial \mathcal{H}}{\partial H}, \quad \dot{H} = -\frac{\partial \mathcal{H}}{\partial h}, \quad (10)
\]

\[
\mathcal{H} = -\frac{1}{4\pi} \ln M_{12} M_{13} M_{14} M_{23} M_{24} M_{34},
\]

where

\[
M_{12} = I - G + 2\sqrt{(1-H)(I-G)} \cosh h,
\]

\[
M_{34} = I - G - 2\sqrt{(1-H)(I-G)} \cosh h,
\]

\[
M_{13} = I + G + 2\sqrt{(1-H)(I-G)} \cos (h + g),
\]

\[
M_{24} = I + G - H - 2\sqrt{(1-H)(I-G)} \cos (h + g),
\]

\[
M_{14} = \frac{H}{2} + 2\sqrt{(1-H)(I-G)} G \cos g,
\]

\[
M_{23} = H - 2\sqrt{(1-I-G)} G \cos g.
\]

**Remark.** These canonical variables have a natural geometric origin related to the representation of the equations of motion on a Lie algebra [6, 7].

**ABSOLUTE MOTION: QUADRATURES AND GEOMETRIC INTERPRETATION**

According to (8), given \( M_{ij}(t) \), we have to know the angles \( \theta_{ij}(t) \) in order to determine the coordinates of the vortices. Clearly, only one of the angles is independent (for example, \( \theta_{12} \)), and the others can be computed by the cosine theorem. The evolution of \( \theta_{ij} \) can be obtained using quadratures [3]:

\[
4\pi \dot{\theta}_{ij} = \frac{2}{M_{ij}} \sum_{k=1}^{n} \Gamma_{k} + \sum_{k \neq i,j}^{n} \Gamma_{k} \left( \frac{1}{M_{ik}} + \frac{1}{M_{jk}} \right) \nonumber - \frac{1}{M_{ij}} \sum_{k \neq i,j}^{n} \Gamma_{k} \left( \frac{M_{ik}}{M_{ik}} + \frac{M_{jk}}{M_{jk}} \right). \quad (11)
\]

For periodic solutions to reduced system (9), (10), there is a remarkable geometric interpretation of absolute motion.

**Proposition 3.** Let \( \gamma(t) \) be a periodic solution (of period \( T \)) to the reduced system. Then, the following is true:

(i) There exists a frame of reference uniformly rotating at an angular velocity \( \Omega_a \) about the center of vorticity, in which each vortex travels along a closed curve \( \xi(t) \).

(ii) \( \Omega_a \) is given by

\[
\Omega_a = \frac{1}{T} \int_{0}^{T} \dot{\theta}_{12}(t) dt \quad (12)
\]

(up to \( \frac{2\pi p}{T q} \), \( p, q \in \mathbb{Z} \)).

(iii) If \( \Omega_a = \frac{p}{q} \) are commensurable (i.e., \( \Omega_a = \Omega_0 \)), then the vortices in a fixed frame also travel along closed curves.

(iv) If some of the curves \( \xi(t) \) can be mapped onto each other by rotation about the center of vorticity through an angle commensurable with \( 2\pi \), then there is a (rotating) frame in which the corresponding vortices move along the same curve.

**Remark.** Proposition 3 can be extended without any change to the arbitrary number \( n \) of vortices if \( \gamma(t) \) is a periodic solution to the reduced system with \( 2n - 2 \) degrees of freedom for the \( n \)-vortex system (for a more detailed discussion on reduction, see, e.g., [1]).

**ANALYTICAL CHOREOGRAPHIES**

Let us now show that the four- and three-vortex systems have remarkable periodic solutions such that all the vortices follow each other on the same curve. Such solutions are referred to as simple choreographies. Choreographies in a fixed and a rotating frame of reference are referred to as absolute and relative, respectively [8].

**Theorem 1** [1]. In the problem of three vortices of equal intensity, if the constants \( I \) and \( \mathcal{H} \) of the integrals
of motion satisfy the inequalities $-\ln 3 < -\frac{4\pi}{3}H + \ln I < \ln 2$, then the motion is a simple relative choreography.

The four-vortex system has a particular solution given by explicit formulas (namely, Goryachev’s solution), in which the vortices form a parallelogram at every instant of time [9]. As in the three-vortex system, the following result can easily be shown to hold true.

**Theorem 2** [1]. *In the problem of four vortices of equal intensities, if the vortices form a centrally symmetric configuration (parallelogram) and the constants $H$ and $I$ satisfy $-\ln 2 < -\frac{4\pi}{3}H + \ln I < -\ln \frac{144}{5}$, then the motion is a simple relative choreography.*

An example of a relative choreography in this case is shown in Fig. 1.

**Remark.** From a geometric viewpoint, the inequalities are interpreted as follows: for a fixed $I$, the type of the motion in the three-vortex system and in the case of Goryachev’s solution changes at the energy values corresponding to Thomson’s and collinear configurations.

### A NEW PERIODIC SOLUTION TO THE FOUR-VORTEX SYSTEM

Let us now show that the four-vortex system has at least one choreography (other than Goryachev’s solution) in addition to that described above. Consider the neighborhood of Thomson’s solution, i.e., configurations in which the vortices are located at the vertices of a square and rotate uniformly around the center of vorticity [10]. Obviously, for two-degree reduced system (10), Thomson’s solution is represented by a fixed point, more precisely, by six points corresponding to the various permutations of the vortices located at the square vertices. Consider one of the permutations with the coordinates $G = 0, H = \frac{1}{2}$, and $h = \frac{\pi}{2}$ (the other permutations are identical). We find the normal form of the Hamiltonian of system (10) in the neighborhood this point. For this purpose, we make a canonical change of variables,

$$G = \frac{x^2 + X^2}{2}, \quad g = \arctan \frac{x}{X},$$

$$H = \frac{1}{2} + \frac{1}{8^{1/4}}Y, \quad h = \frac{\pi}{2} + 8^{1/4}y,$$

and then expand the Hamiltonian in a series up to quadratic terms to obtain

$$H = \frac{\ln 2}{\pi} + H_2 + H_r,$$

where

$$H_2 = \frac{1}{4\pi} (3(x^2 + X^2) + 2\sqrt{2}(y^2 + Y^2)), \quad (13)$$

and the expansion of $H_r$ begins with third-order terms.

Thus, the Hamiltonian $H_2$ defines an integrable system with two incommensurable frequencies and has exactly two nondegenerate periodic solutions at each energy level $H_2 = h_2 = \text{const.}$ These solutions are given by

$$x = X = 0, \quad y^2 + Y^2 = \frac{\pi}{\sqrt{2}}h_2, \quad (14)$$

$$y = Y = 0, \quad x^2 + X^2 = \frac{4\pi}{3}h_2. \quad (15)$$

According to Lyapunov’s theorem [11], these solutions are invariant under perturbations; hence, in the neighborhood of the fixed point, the complete system also has a pair of nondegenerate periodic solutions at each energy level. It is easy to verify that one of the solutions, corresponding to (14), is identical to Goryachev’s solution (i.e., the moving vortices form a parallelogram), whereas the other solution, corresponding to (15), lacks such a simple geometric interpretation.

Since equations of motion (1) are invariant under a cyclic permutation $\sigma_c(z_1, z_2, z_3) = (z_3, z_1, z_2)$ of the vortices and the eigenvalues of $H_2$ are distinct, it is easy to show that both periodic solutions are also invariant under $\sigma_c$. Thus, according to Proposition 3, all the vortices travel along the same curve in a suitable frame; i.e., both solutions correspond to simple relative choreographies. Figure 2 shows an example of a relative choreography corresponding to the new periodic solution of reduced system (10).
RELATIVE AND ABSOLUTE CHOREOGRAPHIES

In general, for each periodic solution (of period $T$) to reduced system (9), (10), there exists a countable set of rotating frames of references in which the vortices move along closed curves. Indeed, if we perform the substitution

$$\Omega_a' = \Omega_a + \frac{p 2\pi}{q T}, \quad p, q \in \mathbb{Z},$$

the trajectories in the corresponding rotating frame remain closed. However, for arbitrary $p$ and $q$, substitution (16) does not preserve connectedness; i.e., in the general case, after the transition to a frame rotating with a frequency $\Omega_a'$, a simple relative choreography splits into separate closed curves, along which the vortices move. Below is a criterion for preserving connectedness.

**Proposition 4.** Suppose that a periodic solution (of period $T$) of the reduced system corresponds to a connected relative choreography in a frame rotating at an angular velocity $\Omega_a$, and let the period of motion of the vortices along the corresponding common curve be equal to $mT$. If

$$mp = knq,$$

where $n$ is the number of vortices and $k \in \mathbb{Z}$ is an arbitrary integer, then transformation (16) gives an connected choreography.

Relation (17) is a sufficient but not necessary condition for a choreography to be connected. If the trajectory of the vortices has additional symmetries, then, in addition to $p$ and $q$ satisfying condition (17), there are more velocities of form (16) that lead to connected choreographies (see the reasoning below for Goryachev’s solution). Interestingly, transformation (16) makes it possible to untwine some choreographies, i.e., eliminate the self-intersections from the curve along which the vortices move.

As was shown above (see Proposition 3), for a relative choreography, if the period $T$ of the solution to the reduced system is commensurable with the rotation period $T_a = \frac{2\pi}{\Omega_a}$ of the frame, then all the vortices move along closed (as a rule, different) curves in the fixed frame.

Let us examine, in more detail, the existence of absolute choreographies in the three- and four-vortex systems. According to what was said above, any relative choreography corresponding to a periodic solution (of period $T$) of the reduced system [see (9) and (10)] becomes closed in the time $mT$, $m \in \mathbb{N}$. During this time interval, the vortices pass through the same relative configuration $m$ times. The angular velocities of the frames corresponding to these choreographies are denoted by $\Omega_m^{(k)}$.

It was shown above that, for fixed $D$ and $E_T < E < E_C$ (where $E_T$ and $E_C$ are the energies corresponding to Thomson’s and collinear configurations), all the solutions to the reduced system for three vortices correspond to connected relative choreographies. Moreover, there is a frame of reference in which any choreography becomes closed in the (minimum possible) time $T$. The corresponding angular velocity is denoted by $\Omega_0^{(0)}$. Its plot is shown in Fig. 3. The angular velocities of the other connected relative choreographies are now given by

$$\Omega_m^{(k)}(E) = \Omega_0^{(0)}(E) + \frac{3k}{m}\Omega_0(E),$$

where $m \in \mathbb{N}$, $k \in \mathbb{Z},$

3$k$ and $m$ are coprime numbers, and $\Omega_0(E) = \frac{2\pi}{T(E)}$. The velocity $\Omega_m^{(k)}$ corresponds to a choreography that becomes closed in the time $mT$.

Obviously, the absolute choreographies are defined by the solutions to the equation $\Omega_m^{(k)}(E) = 0$, where $k$ and $m$ are fixed and $E$ is unknown. Figure 3 shows the graphs of certain $\Omega_m^{(k)}(E)$ and the simplest of the corresponding absolute choreographies. In the general case, there is a countable set of absolute simple choreographies with different $m$ and $k$.

For Goryachev’s solution to the four-vortex system, the line of reasoning is somewhat modified. First, we can show that the simplest connected choreography becomes closed in the time $2T$, and the vortices go through the same relative configuration twice (i.e., the velocities $\Omega_1^{(k)}$ correspond to disconnected choreographies). The graph of one of the corresponding angular velocities, denoted by $\Omega_2^{(0)}$, is shown in Fig. 4. In this

![Fig. 2.](image-url)
case, since the curve corresponding to the simple choreography with $\Omega_2^{(0)}$ is symmetric, the combinations of velocities corresponding to connected choreographies are given by a relation different from (18), namely, by

$$\Omega^{(k)}_{2m}(E) = \Omega_2^{(0)}(E) + \frac{k}{m} \Omega_0(E),$$

where $m$ is odd, $k \in \mathbb{Z}$.

Here, $k$ and $m$ are coprime numbers and $\Omega_0(E) = \frac{2\pi}{T(E)}$, where $T$ is the period of Goryachev’s solution to system (10). This choreography becomes closed in the time $2mT$.

As before, the equation $\Omega^{(k)}_{2m}(E) = 0$ defines absolute simple choreographies. By analogy with the three-vortex system, we can show that there is a countable set of
absolute choreographies corresponding to Goryachev’s solution for different \( m \) and \( k \).

**STABILITY**

To conclude, we discuss the stability of the indicated periodic solutions. Since the three-vortex system is integrable, all the solutions to reduced system (9) are periodic and stable. At the same time, it is easy to show that any (absolute or relative) choreography in this problem is neutrally stable with respect to perturbations in the positions of the vortices in the absolute space (i.e., in a fixed frame of reference).

Since the four-vortex system is nonintegrable, the corresponding relative choreographies may be (exponentially) unstable. At the same time, if a periodic solution to system (10) is stable, then the corresponding choreographies are neutrally stable in the absolute space. A numerical analysis of the multiplicators of the periodic solution corresponding to the choreography displayed in Fig. 4 shows that this solution is (exponentially) unstable.

**REFERENCES**