Invariant Submanifolds of Genus 5 and a Cantor Staircase in the Nonholonomic Model of a Snakeboard

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In this paper, we address the free (uncontrolled) dynamics of a snakeboard consisting of two wheel pairs fastened to a platform. The snakeboard is one of the well-known sports vehicles on which the sportsman executes necessary body movements. From the theoretical point of view, this system is a direct generalization of the classical nonholonomic system of the Chaplygin sleigh. We carry out a topological and qualitative analysis of trajectories of this dynamical system. An important feature of the problem is that the common level set of first integrals is a compact two-dimensional surface of genus 5. We specify conditions under which the reaction forces infinitely increase during motion and the so-called phenomenon of nonholonomic jamming is observed. In this case, the nonholonomic model ceases to work and it is necessary to use more complex mechanical models incorporating sliding, elasticity, etc.

Keywords: Nonholonomic mechanics; snakeboard; qualitative analysis; bifurcations; regularization (blowing up singularities); system on a torus; nonholonomic jamming; bifurcation analysis.

1. Introduction
Snakeboard [1] is a platform to which two movable (free) wheel pairs located at some distance from each other are fastened (see Fig. 1). The dynamics of a snakeboard is usually described using the nonholonomic model, which assumes that there is no slipping in the system. The application of nonholonomic mechanics equations to wheel mechanisms is discussed in the recent paper [Borisov et al., 2015b]. The problem of global dynamics of a snakeboard has received very little attention in the literature, although it is of great practical interest.

[1] Snakeboard is a commercial sports vehicle on which the sportsman executes necessary body movements to achieve a desired direction of motion.
This interest is stimulated by the development of mobile robotics. Most publications on snakeboards are devoted either to numerical simulations of equations of motion or to the search for various “snake-like” (sinusoidal) trajectories arising due to control actions.\cite{Kuleshov2006, Lewis et al. 1994}. Such trajectories are achieved by controlling the rotor and the angles of rotation of the wheel pairs.

As compared to the classical and best-understood problems of nonholonomic mechanics (Chaplygin sleigh, Carathéodory\cite{1933}, Borisov & Mamaev\cite{2002}, Suslov problem; Borisov\cite{2011}) dynamically asymmetric balanced Chaplygin ball\cite{Borisov et al. 2013b, Chaplygin 1903}, the system under consideration is more complicated when it comes to dynamical description, and requires using ideas from topology, bifurcation theory and the theory of singularities.

However, as shown in this paper, free dynamics is regular (albeit nonintegrable) and cannot exhibit chaotic behavior.

The first example of such regular, but nonintegrable behavior in nonholonomic mechanics was considered in Borisov\cite{2012}, Bizyaev\cite{2014} and is related to the rolling motion of a “rubber” unbalanced and dynamically asymmetric ball on a plane. Despite the fact that topologically this problem is equivalent to the Euler case from rigid body dynamics, the dependence of the rotation number on a torus on the values of first integrals is a Cantor staircase (also called Devil’s staircase), which is typical of systems on a torus with limit cycles\cite{Arnold1961, Poincaré1912} and hence without a continuous invariant measure. The fact that there is no continuous invariant measure in the general case for nonholonomic systems was first pointed out in Kozlov\cite{1995} (see also Borisov & Mamaev\cite{2004}). We also mention the papers Bizyaev et al.\cite{2014, 2015}, which discuss the quantization of rotation numbers for the nonholonomic Suslov problem isomorphic to the Hess–Appelrot system. Similar examples occur when the Josephson effect in superconductor physics\cite{1979, 2011, 2012} and in the dynamics of connected oscillators\cite{2012, 2014} are explained.

Another example of a system without a continuous invariant measure which is the most similar to the system considered here is the problem of the motion of a wheel vehicle (platform) with a fixed rear axle. This problem is obtained from the snakeboard problem by introducing an additional holonomic constraint forbidding the rotation of one of the wheel pairs. The study of the motion of this system goes back to\cite{Bottema1964}. In this case, almost all trajectories of the vehicle on a plane are bounded. They are either quasi-periodic or tend asymptotically to a circle. Borisov et al.\cite{2014, 2015}, Borisov et al.\cite{2015a}, Bravo-Doddoli & García-Naranjo\cite{2012}.

In this paper, it is shown that the common level surface of the first integrals of a system describing the free snakeboard dynamics is a sphere with five handles that are glued together along four circles from two tori. A similar example of a system in nonholonomic mechanics in which the level surface of first integrals is a sphere with five handles is given in Hu & Santoprete\cite{2013}. However, in this case, in a neighborhood of the gluing submanifold the velocities and constraint reactions vanish, i.e. they are essential singularities of the flow.

After a regularizing rescaling of time, these four circles become invariant manifolds with two or four equilibrium points lying on them. In the case of two equilibrium points they are of saddle type, and in the case of four equilibrium points we have three saddles and one node. Transitions between these two cases lead to strong rearrangements of the flow. In the first case, we obtain motions tending to a limit cycle, and the analysis reduces to a one-dimensional Poincaré map on a torus with a characteristic Cantor staircase for rotation numbers. In the second case, any trajectory tends to a nonsingular equilibrium point of the system at which the axes of the wheel pairs are parallel to the axis between their centers of attachment. It turns out that this position of the system is impassable because the reaction forces and velocities grow infinitely.

From a mechanical point of view, the nonholonomic model becomes inapplicable for such motions. This phenomenon, which can be called nonholonomic jamming, has not been observed before. Such a
growth of reaction forces can apparently be avoided by incorporating slipping in the presence of dry friction. However, it is unclear whether this is sufficient to eliminate jamming. Note that jamming in systems with dry friction has been known for a long time and is one of the Painlevé paradoxes which are usually described beyond the scope of rigid body dynamics [Pfeifer, 1909].

Thus, the essential feature of global dynamics is due to the complicated topological structure of the phase flow, which also leads to jamming of the typical trajectory. This phenomenon is examined in detail in the conclusion of this paper, and, depending on the initial conditions, the motion either leads to an impact or is impact-free.

Note that for related problems, for example, roller racer dynamics, these phenomena are not observed [Bizyaev, 2017]. In view of this, it is interesting to investigate the controlled dynamics of the system, for example, by considering periodic changes in parameters. For the Chaplygin sleigh such an investigation was carried out in [Borisov et al., 2013a]. The main question in this system is how jamming can be avoided in the control system or, if this is impossible, how a more real model of the system can be constructed. All these questions are of great interest to mobile robotics.

2. Equations of Motion and Conservation Laws

2.1. Definitions and constraints

Consider the problem of the uncontrolled motion of a vehicle with two free wheel pairs which rolls without slipping on a horizontal plane. Assume that the wheel pairs are equal and balanced (i.e. the center of mass of the wheel pair is on the vertical axis passing through the middle of the segment connecting the points of contact of the wheels with the support). Each wheel pair is fastened to the common framework so that it can rotate relative to it about the axis passing through the center of mass of the pair.

We also assume that the wheel pairs are symmetric, i.e. the moment of inertia of the whole vehicle does not depend on the angles of rotation of the pairs [Fig. 2(a)].

As shown in [Borisov et al., 2015a], if the wheel pairs of the vehicle are replaced by skates with suitable characteristics (mass and moment of inertia), then the equations of motion describing the position and orientation of the vehicle and the orientation of the axes of the pair remain unchanged. Therefore, in what follows we shall consider the motion of a sleigh with two rotating skates [see Fig. 2(b)]. The skates are also assumed to be identical, and the vertical axes passing through the points of their attachment to the framework $F_1, F_2$ also pass through the center of mass of the skates and the points of contact of their knife edges with the supporting plane. (If necessary, the rotation of the wheels can be reconstructed by following [Borisov et al., 2013a].)

Let us define two coordinate systems [see Fig. 2(b)]:

- a space-fixed, inertial coordinate system $Oxy$;
- a moving coordinate system $O_1 x_1 x_2$ attached to the framework so that the origin $O_1$ lies on a
straight line joining the axes of rotation of the skates, and the center of mass $C$ lies on the axis $O_1x_1$. 

We specify the position of the vehicle by the coordinates $(x, y)$ of the origin of the moving system $O_1$, and its orientation by the angle of rotation $\psi$ of the moving axes relative to the fixed axes, and denote the angles of rotation of the links relative to the framework by $\theta_1$ and $\theta_2$. Thus, the configuration space of the system is the product of the group of (orientable) motions of the plane and the two-dimensional torus $\mathcal{N} \approx SE(2) \times \mathbb{T}^2$.

In the moving coordinate system $O_1x_1x_2x_3$ the points of attachment of the skates, $P_1$ and $P_2$, and the center of mass, $C$, are given by the radius vectors

$$r_1 = (0, b_1), \quad r_2 = (0, -b_2), \quad R_c = (\eta, 0).$$

The assumption that there is no slipping at the points of contact of the plane with the knife edges of the skates in directions perpendicular to the knife edges can be represented in the form of two non-holonomic constraint equations

$$f_1 = (v_1 - \omega b_1) \cos \theta_1 + v_2 \sin \theta_1 = 0,$$

$$f_2 = (v_1 + \omega b_2) \cos \theta_2 + v_2 \sin \theta_2 = 0,$$

where $\nu = (v_1, v_2)$ is the velocity of the point $O_1$ and $\omega$ is the angular velocity of the framework. Here and in what follows (unless otherwise specified), all vectors are referred to the moving coordinate system $O_1x_1x_2x_3$.

An unusual feature of these constraints is that the system $\mathbb{B}$ degenerates at points of three-dimensional submanifolds in $\mathcal{N}$, which are given by

$$\theta_1 = \theta_2 = \frac{\pi}{2}, \quad \theta_1 = -\theta_2 = \frac{\pi}{2}.$$

This implies that the dimension of subspaces $D_q \subset \mathcal{T}\mathcal{N}_q$, $q \in \mathcal{N}$, given by the constraints $\mathbb{B}$, is not constant everywhere on $\mathcal{N}$ (here $\mathcal{T}\mathcal{N}_q$ denotes the tangent space to $\mathcal{N}$ at point $q$).

For example, everywhere on $\mathcal{N}$ except for submanifolds $\mathbb{D}$, $\dim D_q = 3$, if $q$ belongs to one of the submanifolds $\mathbb{D}$, then $\dim D_q = 4$.

### 2.2. Equations of motion

Let $I$ be the moment of inertia relative to the vertical axis passing through point $O_1$, $m$ the mass of the whole vehicle (including the skates), and $j$ the moment of inertia of each skate relative to the vertical axis passing through the point of attachment. Then the kinetic energy of the vehicle can be written as

$$T = \frac{1}{2} m (v_1^2 + v_2^2) + m v_2 \omega + \frac{1}{2} I \omega^2$$

$$+ j v_2 (\dot{\theta}_1 + \dot{\theta}_2) + \frac{1}{2} j (\dot{\theta}_1^2 + \dot{\theta}_2^2).$$

Let us define the unit vectors normal $(n_1, n_2)$ and tangent $(\tau_1, \tau_2)$ to the plane of the skates, which lie in the horizontal plane [see Fig. 2(b)]. These vectors are expressed in terms of the angles of rotation of the skates as follows:

$$n_k = (\cos \theta_k, \sin \theta_k),$$

$$\tau_k = (-\sin \theta_k, \cos \theta_k), \quad k = 1, 2.$$

To abbreviate some of our forthcoming formulae, we also define the unit vector of the vertical $e_z$ such that its vector product with any horizontal vector $\alpha = (\alpha_1, \alpha_2)$ is written as

$$e_z \times \alpha = -\alpha \times e_z = (-\alpha_2, \alpha_1).$$

The vectors $n_k$ and $\tau_k$ are related by $n_k \times \tau_k = e_z$.

Equations of free motion of a system with undetermined multipliers are written in the form [Borisov et al. 2013, Borisov & Mamaev 2008]:

$$\left( \frac{\partial T}{\partial \dot{q}} \right) + \omega e_z \times \frac{\partial T}{\partial \dot{\omega}} + \sum \lambda_k \frac{\partial f_k}{\partial \dot{q}} = 0,$$

$$\left( \frac{\partial T}{\partial \theta_1} \right) + \nu \frac{\partial T}{\partial \nu} + v_2 \frac{\partial v_1}{\partial \nu} = \sum \lambda_k \frac{\partial f_k}{\partial \theta_1},$$

$$\left( \frac{\partial T}{\partial \theta_2} \right) = 0,$$

where $\nu = \left( \frac{\partial T}{\partial v_1}, \frac{\partial T}{\partial v_2} \right)$. In explicit form these equations are written as

$$\dot{\psi} + \omega e_z \times R_c + \omega e_z \times \nu - \omega^2 R_c = \sum \lambda_k n_k,$$

$$\rho^2 \omega = \sum_{k=1}^{2} \lambda_k (\tau_k \times R_c - \tau_k), \quad \dot{\theta}_k + \dot{\omega} = 0,$$

$$k = 1, 2, \quad \rho^2 = \frac{l - mR_c^2 - 2j}{m}.$$
where the undetermined multipliers \( \lambda = (\lambda_1, \lambda_2) \) are found from the solution of Eqs. \((3)\) together with the time derivatives of the constraints \((1)\), \(f_k = 0\), and have the form

\[
\lambda = \frac{1}{d} \begin{pmatrix}
1 + a_2^2 & \omega_1 a_2 - (n_1, n_2) \\
\omega_2 a_2 - (n_1, n_2) & 1 + a_1^2 \\
\end{pmatrix}
\times \begin{pmatrix}
(\omega + \dot{\theta}_1)(v, \tau_1) + \omega[R_1 + \dot{\theta}_1(n_1, n_1)] \\
(\omega + \dot{\theta}_2)(v, \tau_2) + \omega[R_2 + \dot{\theta}_2(n_1, n_2)] \\
\end{pmatrix}
\]

\[d = \sin^2(\theta_1 - \theta_2) + a_1^2 + a_2^2 - 2a_1a_2 \cos(\theta_1 - \theta_2),\]

\[a_i = \frac{R_i - v_1 \tau_i}{r}.\]

(5)

Equations \((4)\) define the phase flow in the seven-dimensional space \(M^7 = \{(v_1, v_2, \omega, \theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2)\} \) and the constraints \((3)\) are its integrals of motion. The sought-for equations of motion are a restriction of the flow \((4)\) to the five-dimensional invariant manifold

\[M^5_0 = \{(v_1, v_2, \omega, \theta_1, \theta_2) | f_1 = 0, f_2 = 0\} .\]

The equations thus obtained, together with the kinematic relations

\[
x = v_1 \cos \psi - v_2 \sin \psi, \\
y = v_1 \sin \psi + v_2 \cos \psi, \\
\psi = \omega,
\]

form a closed system of equations that describe the free motion of the vehicle.

In this paper, we consider the nonholonomic model of rolling without slipping and neglect friction between the wheels and the plane. We note that it is more convenient to analyze systems with friction if one knows the types of motion of the corresponding system where friction is neglected. As a rule, various types of motion of the original system (without friction) are also qualitatively different when friction is added.

One can incorporate rolling friction into this model (see, e.g., Borisov et al., 2017; Karavaev et al., 2017). However, the forces of rolling friction have generally fairly small values, and the dynamics is little affected by them on small time intervals of motion.

Taking the slipping of wheels and the corresponding forces of sliding friction into account leads to considering a different model which describes the motion of a wheeled vehicle in the sliding mode. That is a challenging problem in its own right. Here we only point out that the phase space has regions in which it is impossible to describe correctly the motion of a wheeled vehicle without taking slipping into account.

Assume that \((\theta_1, \theta_2) \rightarrow \theta,\) where \(\theta\) denotes one of the degeneracy points of the constraints \((3)\).

Then from \((3)\) we find that \(d \rightarrow 0\), i.e. \(\lambda \rightarrow \infty\). Consequently, in a neighborhood of the points \((3)\) the right-hand sides of the equations, i.e. the horizontal components of reaction forces and the reaction torque become arbitrarily large, i.e. the so-called phenomenon of jamming arises. We recall that the reaction forces and the reaction torque are caused by static friction forces at contact points, which cannot exceed some maximal value in real mechanical systems. Consequently, sliding always begins in some neighborhood of the points \((3)\), and the equations of motion \((4)\) must be replaced with equations corresponding to the sliding model.

2.3. Conservation laws

The last pair of Eqs. \((4)\) allows us to find two obvious linear (in velocities) first integrals of the system:

\[C_1 = \theta_1 + \omega = \text{const}, \quad C_2 = \dot{\theta}_2 + \omega = \text{const}.\]

In addition, since the constraints \((3)\) are homogeneous in velocities, the system possesses an energy integral that coincides with the kinetic energy \((5)\). Using the integrals \((4)\), we construct a quadratic integral in a form that is more convenient for further analysis:

\[
E = m^{-1} \left[ T - \frac{1}{2}(\dot{\theta}_1 + \omega)^2 + (\dot{\theta}_2 + \omega)^2 \right] = \frac{1}{2}(v + e_3 \times R_1 \omega)^2 + \frac{1}{2}R_2 \omega^2 = \text{const}.\]

(8)

Moreover, the complete system of Eqs. \((4)\) and \((5)\) possess obvious symmetry fields

\[
u_1 = \frac{\partial}{\partial x}, \quad \nu_2 = \frac{\partial}{\partial y}, \quad \nu_3 = \frac{\partial}{\partial \psi}.
\]

Due to these symmetry fields, Eqs. \((4)\) decouple from the general system (for a theoretical analysis, see Borisov & Mamaev, 2014). Note that in the general case the system under consideration possesses no invariant measure (with analytic density). Obstructions to its existence are identified in Sec. 3.
3. Integral Submanifolds

As shown above, from the general system one can decouple Eqs. (3) governing the evolution of the variables \( \theta_1, \theta_2, \psi_1, \psi_2, \omega, \theta_1 \) and \( \theta_2 \), which parameterize the seven-dimensional manifold \( \mathcal{M}^7 \approx \mathbb{T}^2 \times \mathbb{R}^5 \). The integral invariant submanifolds of this system turn out to be two-dimensional and are given by the integrals (4) and the constraints (5). Rewriting them here by making the change of variables

\[
\xi_1 = \frac{\psi_1 - b_1 \omega}{b_1 + b_2}, \quad \xi_2 = \frac{\psi_1 + b_2 \omega}{b_1 + b_2}, \quad \xi_3 = \frac{\psi_2}{b_1 + b_2},
\]

we obtain

\[
\dot{\theta}_1 - \dot{\xi}_1 = c_1, \quad \dot{\theta}_2 + \dot{\xi}_2 = c_2, \quad \dot{\psi}_1 = 0, \quad \dot{\psi}_2 = 0, \quad \dot{\omega} = 0.
\]

Let \( m_0, I_0 \) and \( \mathbf{R}_0 = (\xi_0, \eta_0) \) be the mass, the central moment of inertia and the radius vector of the center of mass of the frame, respectively, and let \( m_1 \) be the mass of each of the wheel pairs. Then, using the definition of the projection of the radius vector of the center of mass of the whole vehicle onto the axis \( \xi_1, \xi_2 \), \( m_0 \xi_1 = m_1 \eta_0 \), it is easy to show that

\[
\rho^2 = \frac{I_0 + m_0 \xi_0^2}{m_0 (b_1 + b_2)^2} + \frac{m_1 (b_1^2 + b_2^2)}{m_0} + \frac{2 m_1}{m_0} \eta^2.
\]

It follows from (11) and (12) that the new parameters satisfy the relations

\[
\rho^2 > \frac{2 m_1}{m_0} \eta^2, \quad b_1 + b_2 = 1, \quad 0 \leq b_i \leq 1, \quad i = 1, 2.
\]

3.1. Topological type of integral submanifolds

We first note that the first two equations of (10a) define in \( \mathcal{M}^7 \) a family of five-dimensional submanifolds

\[
\mathcal{M}^5_{c,h} = \{ \mathbf{z} | (\theta_1, \theta_2, \xi_1, \xi_2, \xi_3) \} \approx \mathbb{T}^2 \times \mathbb{R}^3,
\]

in which the remaining three equations (10b) and (10c) give rise to two-dimensional surfaces \( \mathcal{M}^2_{c,h} \subset \mathcal{M}^5_{c,h} \).

Proposition 1. The integral submanifolds \( \mathcal{M}^2_{c,h} \) of the system (3), which are given by (11) and (14), are a smooth two-dimensional surface of genus 5 (a sphere with five handles) when \( h \neq 0 \).

Proof. We first prove that the above surface contains no singularities. For this, we calculate the \( 3 \times 5 \) matrix

\[
\frac{\partial (f_1, f_2, E)}{\partial \mathbf{z}} = \begin{pmatrix}
-\xi_1 \sin \theta_1 + \xi_3 \cos \theta_2 & 0 & 0 \\
0 & -\xi_2 \sin \theta_2 + \xi_3 \cos \theta_1 & 0 \\
\mathbf{e}_1^T & \mathbf{e}_2^T & \mathbf{e}_3^T
\end{pmatrix},
\]

where \( \mathbf{A} \) is the Hessian of the integral in the variables \( \xi \). We see that when \( \xi \neq 0 \), the rank of this matrix is equal to three. Thus, the manifold under consideration contains no singularities.

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*It is easy to check that on the submanifold \( \mathcal{M}^5_{c,h} \) there exist sets of points in which the rank of the matrix \( [14] \) falls (for example, \( \xi_1 = \xi_2 = 0, \theta_1 = \theta_2 = \frac{\pi}{2} \)). However, these points do not belong to the integral manifold \( \mathcal{M}^2_{c,h} \) at arbitrary \( h \neq 0 \) everywhere on \( \mathcal{M}^2_{c,h} \).*
We now consider Eqs. (10a) and (10b) as a system of three equations in $\xi$ with fixed values $\theta_1$ and $\theta_2$, and represent it as
\[(\xi, e_1) = 0, \quad (\xi, e_2) = 0, \quad (\xi, A\xi) = 2h.\]
Consequently, the sought-for vector $\xi$ is perpendicular to the vectors $e_1$ and $e_2$, and its end lies on the surface of the ellipsoid given by the third equation of (10).

The vectors $e_1$ and $e_2$ are not parallel everywhere on $\Sigma^3 = \{ (\theta_1, \theta_2) \mod 2\pi \}$ except for the points
\[\theta_1 = \theta_2 = \frac{\pi}{2}, \quad \theta_1 = \theta_2 = -\frac{\pi}{2}, \quad \theta_1 = -\theta_2 = \frac{\pi}{2}, \quad \theta_1 = -\theta_2 = -\frac{\pi}{2} \tag{16}\]
In the case where the vectors $e_1$ and $e_2$ are not parallel to each other, the system (10) has two solutions $\xi_+, \xi_-$, which differ in the orientation of the trihedron $e_1, e_2, \xi$.

Thus, the surface $M^2_{c,h}$ is a pair of two-dimensional tori $M^2_{e_1}, M^2_{e_2}$ with four excluded points glued together along some boundary $S\varepsilon$, which is projected into points (16) on the torus $\Sigma^2 = \{ (\theta_1, \theta_2) \}$.

To give a formal description of the sets $M^2_{\xi, \epsilon}, S\varepsilon$, we define on the manifold a smooth function
\[\Delta = (e_1 \times e_2, \xi)|_{M^2_{\epsilon, \xi}}, \tag{17}\]
which is the oriented volume of the parallelepiped spanned by the vectors $e_1, e_2$, and $\xi$. This function (with $h \neq 0$) possesses the following properties:

1. On the subset $M^2_{\epsilon}$, the function $\Delta$ is strictly positive, while on $M^2_{\xi}$ it is strictly negative;
2. $\Delta = 0$ if and only if $e_1 \parallel e_2$; hence, the boundary $S\varepsilon$ coincides with the set of zeros of $\Delta$;
3. $\Delta$ has no critical points on $S\varepsilon$.

The first two properties follow from Eqs. (10a) and (10b) and from the definition of the vectors $e_1$, $e_2$, $\xi_+$, $\xi_-$, and the set $S\varepsilon$. The last property can be checked by an immediate calculation of the rank of the matrix $\frac{\partial}{\partial \theta_1, \partial \theta_2, \partial \xi} \Delta$ at points (16). This rank turns out to be $3$ (for $\xi \neq 0$).

Consequently, the integral submanifold $M^2_{c,h}$ is a union $M^2_{\epsilon} \cup M^2_{\xi} \cup \Sigma^2$ where
\[M^2_{\epsilon} = \{ z | z \in M^2_{c,h}, \Delta > 0 \}, \quad M^2_{\xi} = \{ z | z \in M^2_{c,h}, \Delta < 0 \}, \quad \Sigma^2 = \{ z | z \in M^2_{c,h}, \Delta = 0 \}.
\]

Both parts $M^2_{\epsilon}$ and $M^2_{\xi}$ are glued together along the boundary $S\varepsilon$, which coincides with the above-mentioned hyperplane $\Delta = 0$ on $\Sigma^2$. It follows from (10a) and (10b) that the submanifold $S\varepsilon$ is a union of four circles that are projected in $\mathbb{R}^3 = \{ \xi \}$ into the same ellipse given by $\xi_1 = \xi_2 = \xi_3 = 0$.

Thus, the integral submanifold consists of two tori glued together along four circles, see Fig. 3a. This surface is a surface of genus 5 (i.e. a sphere with five handles).

Interestingly, the topology of integral manifolds in the problem we consider here does not depend on the values of first integrals when $h \neq 0$. In other words, the phase space (16) with the excluded submanifold $h = 0$ (i.e. the torus $\Sigma^2$ given by $\xi_1 = \xi_2 = \xi_3 = 0$) is trivially foliated by integral submanifolds.

3.2. Visualization of an integral submanifold

(i) To visualize the integral submanifold $M^2_{c,h}$, we consider the projection of $M^2_{c,h}$ onto the three-dimensional torus $\mathbb{R}^3 = \{ (\theta_1, \theta_2, \varphi) \mod 2\pi \}$, where $\varphi$ is the angular coordinate for cylindrical coordinates in the space $\mathbb{R}^3 = \{ \xi \}$:
\[\xi_1 = r \cos \varphi, \quad \xi_2 = r \sin \varphi, \quad \xi_3 = z. \tag{18}\]

The image of the integral submanifold $M^2_{c,h}$ with such a projection $p_\varphi(M^2_{c,h})$ is a two-dimensional surface in $T^3$. From (10) we find that this
surface is given by the trigonometric relation

$$\sin \theta_2 \cos \theta_1 \cos \varphi - \cos \theta_2 \sin \theta_1 \sin \varphi = 0.$$ (19)

Note that $M^2_{c,h}$ is projected onto $T^3$ uniquely and without singularities.

The surface given by (19) is shown in Fig. 4(a). Due to the symmetry of (19) relative to the shift of any angle $\theta_1$, $\theta_2$, and $\varphi$ by $\pi$, the corresponding surface consists of eight equal parts, one of which is depicted in Fig. 4(b). The figure shows that in a neighborhood of the circles given by (16), the surface is similar to the surface of a helicoid. Figures 4(c) and 4(d) show projections of the submanifolds $M^2_+$ and $M^2_-$. The heavy lines in Fig. 4 represent the gluing submanifold $S$, which in the chosen projection consists of four vertical straight lines.

(ii) Note that for the study of phase flows on two-dimensional integral submanifolds, it is important to correctly choose a suitable projection of phase space. Otherwise, self-intersections and singularities of a projection of Whitney umbrella type [Borisov et al., 2010] can arise on the manifolds (more precisely, on their projections). We recall that a surface given in some coordinates by the equation $x^2z = y^2$ [Fig. 5(a)] is called a Whitney umbrella.

Indeed, by expressing $v$ from the constraint equations and substituting them into (16), we
obtain an equation for the projection of the integral manifold onto the space \( \{(\theta_1, \theta_2, \omega)\} \). In a small neighborhood of one of the points in (16), \( \theta_1 = \theta_2 = \frac{\pi}{2} \), this equation can be represented as

\[
\left( \theta_1 - \frac{\pi}{2} \right) \tilde{b}_1 + \left( \theta_2 - \frac{\pi}{2} \right) \tilde{b}_2 \right)^2 = \left( \frac{2h}{\omega^2} - \tilde{\beta}^2 - \tilde{\eta}^2 \right) (\theta_1 - \theta_2)^2. \tag{20}
\]

It is easy to notice that after the change of coordinates

\[
x = \theta_1 - \theta_2, \quad y = \left( \theta_1 - \frac{\pi}{2} \right) \tilde{b}_1 + \left( \theta_2 - \frac{\pi}{2} \right) \tilde{b}_2, \\
z = \frac{2h}{\omega^2} - \tilde{\beta}^2 - \tilde{\eta}^2,
\]

Eq. (20) reduces to a form that is standard for a Whitney umbrella. Figures 5(b) and 5(c) show corresponding parts of the projection of the integral manifold, on which two Whitney umbrellas can be seen well. The heavy lines in these figures represent self-intersections of the surfaces.

4. Analysis of the Phase Flow

4.1. Restriction of the phase flow

We first note that, according to (5), the undetermined multipliers \( \lambda_1 \) and \( \lambda_2 \) have singularities on the gluing submanifold \( S \) (they go to infinity for \( \cos \theta_1 = \cos \theta_2 = 0 \)). Consequently, the phase flow has singularities (is not defined) on the gluing submanifold \( S \). Thus, we need to investigate
the system’s trajectories separately on two disjoint parts $M^e_2$, $M^n_2$ of the integral submanifold $M^c_2$. On each of the above-mentioned submanifolds the phase flow is smooth. Moreover, due to the symmetry of the wheel pairs (skates) both sets $M^e_2$ and $M^n_2$, and the flow on them are equivalent, and so we consider the projection of the phase flow on one of them, for example, $M^e_2$, onto the torus $\mathbb{T}^2 = \{(\theta_1, \theta_2) \mod 2\pi\}$ with excluded points (16).

Remark 1. The projection of the phase flow on $M^e_2$ onto the same torus can be obtained from the projection of the flow on $M^n_2$ by a shift along one of the coordinates on $\pi$, for example, $\theta_1 \to \theta_1 + \pi$.

As stated in the proof of the theorem, the vectors $e_1$ and $e_2$ are noncollinear at nonsingular points [everywhere except for points (16)], and so, to restrict the system to the constraints (10), we choose

$$\xi = \mu e_1 \times e_2$$

$$= (\mu \sin \theta_1 \cos \theta_2, -\mu \cos \theta_1 \sin \theta_2,$$

$$\mu \cos \theta_1 \cos \theta_2).$$

From (9) we conclude that the value of $\mu$ is related to the angular velocity of the platform by

$$\dot{\xi}_2 - \xi_1 = \omega = \mu \sin(\theta_1 - \theta_2)$$

and the sign $\mu$ coincides with the sign of $\Delta$ defined by (10).

On a fixed level set of the integral (10), the value of $\mu$ is expressed in terms of the variables $\theta_1$, $\theta_2$:

$$\mu^2(\theta_1, \theta_2) = 2h((\dot{\theta}_1^2 + \dot{\theta}_2^2 - \dot{b}_1 \dot{b}_2 \sin^2(\theta_1 - \theta_2)$$

$$+ \dot{b}_1 \cos^2 \theta_1 + \dot{b}_2 \cos^2 \theta_2 + 2\dot{\theta}_1 \cos \theta_1$$

$$\times \cos \theta_2 \sin(\theta_1 - \theta_2)))^{-1}.\quad (22)$$

On the sets $M^e_2$ and $M^n_2$, we need to take the positive and negative solutions of Eq. (22), respectively.

It follows from (10) that for fixed $c_1$, $c_2$, $h$ the evolution of $\theta_1$ and $\theta_2$ is described by

$$\dot{\theta}_1 = c_1 - \sin(\theta_1 - \theta_2)\mu(\theta_1, \theta_2),$$

$$\dot{\theta}_2 = c_2 - \sin(\theta_1 - \theta_2)\mu(\theta_1, \theta_2).\quad (23)$$

Thus, although all integral submanifolds are isomorphic, the flows on them are not equivalent for different values of constant integrals.

Note that the system (22) is invariant under the transformations

$$\theta_1 \to \theta_1 + \pi n, \quad \theta_2 \to \theta_2 + \pi(n + 2k), \quad n, k \in \mathbb{Z}.$$

This symmetry reflects the physical equivalence of the vehicle’s positions when both wheel pairs simultaneously rotate by $180^\circ$.

Figure 6 shows examples of phase trajectories on the plane $(\theta_1, \theta_2)$ for different values of the
and the lower half ($R < 0$) is mapped exactly onto the submanifold (25), and the circle $\{\lambda = 0\}$ is mapped identically. We find the positive definite function similar to (18). The evolution of the variables $R$ and $\Phi$ in a neighborhood of the submanifold (25) is described by

$$RR = R(c_0 \sin(\Phi - \Phi_0) - r(\Phi)(\sin^2 \Phi - \cos^2 \Phi)) + O(R^2),$$
$$R\Phi = c_0 \cos(\Phi - \Phi_0) + r(\Phi)(\sin \Phi - \cos \Phi)^2 + O(R),$$
$$r(\Phi) = \sqrt{\frac{2h}{g(\Phi)}},$$
$$g(\Phi) = (\tilde{b}_1^2 + \tilde{b}_2^2 - b_1 b_2)(\sin \Phi - \cos \Phi)^2 + b_1 \sin^2 \Phi + b_2 \cos^2 \Phi,$$

(29)

where the constants $c_0$ and $\Phi_0$ are related to the constants of the first integrals $c_1$ and $c_2$ by

$$c_1 = c_0 \cos \Phi_0, \quad c_2 = c_0 \sin \Phi_0.$$  

Near the point $\theta_1 = \theta_2 = -\frac{\pi}{2}$, a parameterization similar to (25) has the form

$$\cos \theta_1 = R \sin \Phi,$$
$$\cos \theta_2 = R \cos \Phi,$$

$$\sin \theta_1 = -\sqrt{1 - R^2 \sin^2 \Phi},$$
$$\sin \theta_2 = -\sqrt{1 - R^2 \cos^2 \Phi},$$

and leads to equations that are completely identical to (29). Near the points $\theta_1 = -\theta_2 = \pm \frac{\pi}{2}$, the parameterization has the form

$$\cos \theta_1 = \pm R \sin \Phi,$$
$$\cos \theta_2 = \mp R \cos \Phi.$$
\[ \sin \theta_1 = \pm \sqrt{1 - R^2 \sin^2 \Phi}, \]
\[ \sin \theta_2 = \pm \sqrt{1 - R^2 \cos^2 \Phi}. \]

In this case, the equations for \( R \) and \( \Phi \) are identical to (28) up to time reversal \( t \to -t \). Consequently, the behavior of trajectories in neighborhoods of all points \( \Phi \) is the same. Therefore, in what follows we will consider a solution only near the point \( \Phi \).

We now rescale time in (29) as \( R \to -R \) and obtain thereby a regularized system whose trajectories coincide with the trajectories of the initial system, but in which the singularity (infinite singularity) has been eliminated. Neglecting higher-order terms, we obtain a truncated system describing the vector field in a neighborhood of a singular submanifold:
\[
\begin{align*}
\frac{dR}{dt} &= c_0 \sin(\Phi - \Phi_0) - r(\Phi)(\sin^2 \Phi - \cos^2 \Phi)), \\
\frac{d\Phi}{dt} &= c_0 \cos(\Phi - \Phi_0) + r(\Phi)(\sin \Phi - \cos \Phi)^2.
\end{align*}
\] (30)

4.3. Bifurcation analysis

We now consider in more detail how the behavior of the trajectories of the truncated system (30) and the complete system change depending on the parameters. Since the system is invariant under the change of variable \( R \to -R \), it suffices to consider its dynamics for \( R \geq 0 \).

We first note that (in a neighborhood of \( R = 0 \)) the fixed points of the system (30) coincide with the fixed points of regularization of the complete system (28) and are given by
\[ R = 0, \]
\[ F(\Phi) = c_0 \cos(\Phi - \Phi_0) + r(\Phi)(\sin \Phi - \cos \Phi)^2 = 0. \]

It also follows from (28) that the eigenvalues of a system linearized in a neighborhood of fixed points are always real. Consequently, according to [Hirsch, Pugh...], these fixed points are hyperbolic and the flow in their neighborhood is topologically conjugate to the flow of the linearized system.

Moreover, it follows from (28) that the following invariant submanifolds of the truncated system pass through the fixed points:

- a circle that is given by
\[ R = 0, \] (31)
- straight lines given by
\[ F(\Phi) = 0. \] (32)

Using the transformation \( \tan(\Phi - \pi/4) = z \), one can show that, depending on the values of the parameters and first integrals, Eq. (32) can have two or four roots.

On the plane of first integrals \((c_0, \Phi_0)\), regions with different numbers of roots are separated by bifurcation curves. Equations of these curves can be obtained from the condition for degeneracy of the roots of Eq. (32):
\[ F(\Phi) = 0, \quad \frac{dF}{d\Phi}(\Phi) = 0. \] (33)

Equations (33) define on the plane \((c_0, \Phi_0)\) bifurcation curves which can be represented in parametric form
\[ \Phi_0 = \Phi + \arcsin \frac{-G(\Phi)}{\sqrt{1 + G(\Phi)^2}} - \pi, \]
\[ h = \frac{c_0}{g(\Phi)} \]
\[ G(\Phi) = 2(\cos \Phi + \sin \Phi) \frac{-G(\Phi)}{\cos \Phi - \sin \Phi)^2} + g(\Phi)^2, \quad \Phi \in [-\pi, \pi], \]
where \( g(\Phi) \) was defined in (28).

Figure 7 shows a bifurcation diagram on the plane of first integrals \((c_0, \Phi_0)\) with parameters (33). Gray denotes regions corresponding to the existence of four fixed points of the system (30).
Fig. 8. Phase portraits of the system (30) with parameters (24): (a) for the case of two fixed points with $c_1 = 1.85$, $c_2 = 0.85$, $h = 3$ and (b) for the case of four fixed points with $c_1 = 0.660$, $c_2 = 0.751$, $h = 0.465$.

and the related invariant submanifolds, and white denotes regions with two fixed points and the related invariant submanifolds.

Figure 8 shows phase portraits of the system that correspond to the cases of existence of two [Fig. 8(a)] and four [Fig. 8(b)] fixed points on the circle $R = 0$. Figure 9 gives a more detailed illustration of the process of bifurcation when the bifurcation curve is crossed.

We now consider these cases in more detail.

(i) Cases of Two Fixed Points (Region $A$).

In this case, both fixed points are saddle points, and for the truncated system their separatrices form invariant submanifolds (22) and (31). In the complete system (23), after regularization, the pair of separatrices also lie on the invariant submanifold $R = 0$, the other separatrices are tangent to the submanifolds (31) at fixed points. As seen from Fig. 9(a), the system (23) has only one trajectory (for each singular point), which goes exactly into a singular point along the entering separatrix.

As shown above, in the case of the projection $M^2_{c,h} \rightarrow \mathbb{T}^2 = \{(\theta_1, \theta_2)\}$, four components of the submanifold $S$ are projected into points $\mathbb{R}$, so that the circles in Figs. 8 and 9 shrink to points. The initial flow on the submanifold $M^2_{c,h}$ (and hence $M^2$) is projected into the flow $\mathbb{R}$ on the torus $\mathbb{T}^2 = \{(\theta_1, \theta_2)\}$ with four singular points, in each of which one entering trajectory and one exiting trajectory converge, and the tangents to them at the fixed point do not coincide in the general case.

Thus, we find that there exists a trajectory isomorphism between the flow on the submanifold $M^2_{c,h}$ supplemented with points $\mathbb{R}$, and the flow without fixed points on a two-dimensional torus, which loses smoothness at four points.

A similar statement is valid for $M^2$.

As is well known, the rotation number in this case an invariant that characterizes the flow on $\mathbb{T}^2$.

$$\nu = \lim_{t \to \infty} \frac{\theta_1(t)}{\theta_2(t)} \quad (35)$$

Since the system has no invariant measure (with analytic density), the dependence of the rotation number on the system parameters is a Cantor staircase whose horizontal segments correspond to limit cycles. Figure 10 illustrates successively scaled up fragments of dependence of the rotation number on the value of the integral $h$.

Remark 2. A more traditional representation of the rotation number has the form

$$\nu = \lim_{t \to \infty} \frac{\theta_1(t)}{\theta_2(t)}$$

In this case, the numbers $\nu$ and $\nu'$ are uniquely related by the equation $\nu'^{-1} = 1 - \nu^{-1}$, and this is why we have chosen the representation (35), which has a clearer geometric interpretation.

Phase portraits corresponding to this case are shown in Fig. 8. In this case, the asymptotic attraction of the trajectory to one of the limit cycles is a generic motion. Thus, if the limit cycles are far from the singular points, then there is a set of trajectories of nonzero measure for which the free vehicle moves without jamming.

(ii) Case of Four Fixed Points (Region $B$).

During a transition from region $A$ to region $B$ (see
Fig. 9. (a)–(c) Phase portraits of the truncated system (30), (d)–(f) their representation in the coordinates $x = (R + 1)\sin\Phi$, $y = (R + 1)\cos\Phi$ and (g)–(i) the phase portrait of the complete system (23) in a neighborhood of the singular point (25) with parameters (24) and $c_1 = 0.660$, $c_2 = 0.751$. Figures (a), (d) and (g) correspond to the existence of two fixed points of the system (30) at $h = 0.835$. Figures (b), (e) and (h) correspond to values of first integrals on the bifurcation curve (34) at $h \approx 0.841$. Figures (c), (f) and (i) correspond to the existence of four fixed points of the system (23) at $h = 0.845$.

Fig. 9, an additional pair of fixed points are born in the truncated system (30) and in the regularized system (29). One of these fixed points is a saddle and the other is a node [Figs. (c) and (f)]. It follows from the results of Sec. 4.2 that for one pair of the components of the set $S$, the node is stable (an attractor), and for the second pair it is unstable (a repeller). This implies that after projection onto the torus $\mathbb{T}^2 = \{ (\theta_1, \theta_2) \}$ we obtain a flow with four fixed points, of which two are repellers and two are attractors. Moreover, the trajectories of the system (29) which are not attracted by these attractors either are absent or form a set of zero measure (although it is everywhere dense). As a result, the attraction of the trajectory to a singular point in finite time is a generic motion. In this case one needs
to consider not individual trajectories of the phase portrait, but regions of attraction of different singular points and their bifurcations. An example of partitioning of the phase plane \((\theta_1, \theta_2)\) into regions of attraction of various singular points is presented in Fig. 11.

Thus, in this case all trajectories of the system \((23)\) are at some instant of time in a neighborhood of the singular points \((2)\) where jamming occurs.

### 4.4. Reconstruction

We now discuss briefly the problem of analyzing the evolution of the trajectory and the orientation of the platform. For a given solution \(\theta_1(t), \theta_2(t)\) of the reduced system \((23)\) we find velocities in the form

\[
\begin{align*}
  v_1 &= -\mu(\theta_1, \theta_2)(b_2 \sin \theta_1 \cos \theta_2 + b_1 \cos \theta_1 \sin \theta_2), \\
  v_2 &= (b_1 + b_2)\mu(\theta_1, \theta_2) \cos \theta_1 \cos \theta_2, \\
  \omega &= \mu(\theta_1, \theta_2) \sin(\theta_1 - \theta_2),
\end{align*}
\]

where \(\mu(\theta_1, \theta_2)\) is given by \((22)\). The position \((x, y)\) and the orientation of the snakeboard are now given by the quadratures \((6)\). It is convenient to represent these quadratures in the canonical form

\[
\dot{\psi} = \omega(\theta_1, \theta_2),
\]

\[
\dot{x} + iy = (v_1(\theta_1, \theta_2) + iv_2(\theta_1, \theta_2))e^{i\psi}.
\]

In the case of a periodic and quasi-periodic trajectory of the reduced system \(\theta_1(t), \theta_2(t)\), the quadratures can be analyzed in a standard way by...
Fig. 11. Regions of attraction of singular points for $c_1 = 0.660$, $c_2 = 0.751$, $h = 0.405$. The gray region corresponds to the region of attraction of the point $\theta_1 = \theta_2 = \pi$. The white region corresponds to the region of attraction of the point $\theta_1 = \theta_2 = -\pi$.

Fig. 12. (a) Trajectory of the wheel vehicle in the case of quasi-periodic behavior of the reduced system and (b) a regime with limit cycles, corresponding to the trajectories depicted in Figs. 6(a) and 6(b), respectively.
shows an interesting dynamical effect of the vehicle’s motion when it changes fairly fast from an unstable quasi-periodic trajectory to a stable one.

We now consider dynamical effects observed when the trajectory passes near singular points. It follows from (5) that the undetermined multipliers (constraint reactions) $\lambda$ grow infinitely as these points are approached. In this case, the model of rolling without slipping does not work any more, which makes it necessary to consider a system incorporating slipping and friction forces.

In the case where the trajectory passes near a singular point, but insufficiently close to it for slipping to arise, the system exhibits interesting dynamical effects similar to an impact. Using the second equation of (4), we can show that such a motion undergoes abrupt changes in angular velocity $\omega$. The condition for the integrals (7) to be constant implies that the angular velocities of the wheel pairs $\dot{\theta}_1$ and $\dot{\theta}_2$ also change abruptly. As computer simulations show, the angular velocities can change sign during such a motion. Motion in absolute space

![Fig. 13. (a), (c) Examples of trajectories of point $O_1$ of the vehicle, and (b), (d) the corresponding time dependences of angular velocity $\omega$. Figures (a) and (b) correspond to the parameters $\epsilon_1 = 1.85$, $c_2 = 0.85$, $h = 3$, and to the initial conditions $\theta_1(0) = 0$, $\theta_2(0) = 2$. Figures (c) and (d) correspond to the parameters $\epsilon_1 = 0.660$, $c_2 = 0.751$, $h = 0.405$, and to the initial conditions $\theta_1(0) = \theta_2(0) = 0.$]
in this case is similar to an impact of the vehicle against an elastic wall. An example of trajectories of the vehicle and the corresponding dependences of angular velocity $\omega(t)$ are shown in Fig. 16.

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Invariant Submanifolds of Genus 5 and a Cantor Staircase in the Nonholonomic Model


