LIBRATION POINTS IN SPACES $S^2$ AND $L^2$

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We consider two-body problem and restricted three-body problem in spaces $S^2$ and $L^2$. For two-body problem we have showed the absence of exponential instability of particular solutions relevant to roundabout motion on the plane. New libration points are found, and the dependence of their positions on parameters of a system is explored. The regions of existence of libration points in space of parameters were constructed. Basing on a examination of the Hill's regions we found the qualitative estimation of stability of libration points was produced.

1. Particular solutions of two-body problem

1.1. Introduction

Let us consider the problem of two bodies in constant curvature spaces $S^3 (L^3)$, when the bodies move in some potential field $U(q_1, q_2)$ ($q_1, q_2$ are coordinates of points on $S^3 (L^3)$). In particular, the potential energy $U$ depends on mutual distance (measured along geodesic) between two points. We consider below the generalization of Newtonian potential for space $S^3 (L^3)$. It is obtained from solution of the Laplace–Beltrami equation for curved space [6, 4]. For $S^3$ the potential $V = \gamma \cot \theta$, where $\theta$ is angular distance between bodies $\gamma$, is positive constant of interaction. For $L^3 V = \gamma \coth \theta$, where $\theta$ now is “hyperbolic” angular distance, defined by equation $\langle q_1, q_2 \rangle = \cosh \theta$.

In contrast to the plane, two-body problem is not integrable for generalized Newtonian interaction in curved space [8]. The straightforward study of spatial two-body problem is rather complicated, therefore it is natural (similarly to the plane case) to study the dynamics on the invariant submanifolds of system. The study of system on an invariant submanifold (for example, its nonintegrability and stochasticity on submanifold) allows to make the conclusions valid for all phase space. Whereas in the $n$-body problem on $\mathbb{R}^3$ the invariant manifolds are planes, for the $n$-body problem in the curved space they are spheres $S^2$ (pseudospheres $L^2$).

1.2. Particular solutions of two-body problem on $S^2$ and $L^2$

Let us consider a set of particular solutions of two-body problem on $S^2 (L^2)$. These solutions are analogs of relative equilibriums in the $n$-body problem on a plane [1]. Such solutions for $n = 2$ are given in paper [3].

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Let us consider at first the relative equilibrium in two-body problem on $S^2$. The stationary configurations are critical points of an effective potential

$$ U_{\text{eff}} = -\frac{1}{2}(m_1 \sin^2 \theta_1 + m_2 \sin^2 \theta_2)\omega^2 R^2 - m_1m_2 \cot \alpha, \quad \text{(1.1)} $$

where $\alpha$ is a minimal angle between bodies on geodesic. These stationary configurations are the rotations of two bodies around an axis $\omega$. Both bodies are situated in one plane with rotational spin axis, but in different half-planes relatively to the rotational axis (Fig. 1).

The parameters of orbits are given by

$$ \sin \theta_1 \cos \theta_1 \omega^2 R^2 - \gamma \frac{m_2}{\sin^2 \alpha} = 0, $$
$$ \sin \theta_2 \cos \theta_2 \omega^2 R^2 - \gamma \frac{m_1}{\sin^2 \alpha} = 0. \quad \text{(1.2)} $$

Thus, setting distance between bodies, we can find angles $\theta_1, \theta_2$ and angular velocity, of rotational motion of the bodies so that $\alpha$ remains constant. For any parameters of the system holds $\theta_1 < \pi/2$, $\theta_2 < \pi/2$. The case of space $L^2$ can be considered similarly, with the replacement of trigonometrical functions by hyperbolic ones. As result we obtain

$$ \sinh \theta_1 \cosh \theta_1 \omega^2 R^2 - \gamma \frac{m_2}{\sinh^2 \alpha} = 0, $$
$$ \sinh \theta_2 \cosh \theta_2 \omega^2 R^2 - \gamma \frac{m_1}{\sinh^2 \alpha} = 0. \quad \text{(1.3)} $$

On $S^2$ for $\alpha < \pi/2$ the more massive body moves on smaller radius and its position tends to the north pole with decreasing of the ratio $\frac{m}{M}$. This curvature configuration turns to the usual rotational motion of two bodies on the plane with decreasing. At large mutual distances $\alpha > \pi/2$ the more massive body moves along the greater orbit and turns to motion along geodesic (in this case along equator) with decrease of the ratio $\frac{m}{M}$. This type of motion does not correspond to any configuration in the plane case.

Let us consider small deviation of the system from the case of equal masses: $m_1 = m$, $m_2 = m + \delta m$. From the system (1.2) one can derive in the first order in $\delta m$:

$$ \theta_1 = \frac{\alpha}{2} + \delta \theta, \quad \theta_2 = -\frac{\alpha}{2} + \delta \theta, \quad \delta \theta = \frac{\tan \alpha \delta m}{4m}. \quad \text{(1.4)} $$

Thus, indeed, for $\alpha < \pi/2$ the inequality $\delta \theta > 0$ fulfilled and the position of massive body tends to the rotational axis, for $\alpha > \pi/2$ the inequality $\delta \theta < 0$ fulfilled and the position of massive body tends to the equator.

For $L^2$ the equality $\delta \theta = \frac{\tan \alpha \delta m}{4m}$ is valid, and therefore only the first type of motions exists (similar to plane motions, and turn into plane motions when the curvature vanishes).

1.3. Restricted two-body problem

When the mass of one of bodies tends to infinity, and the interaction energy remains finite, there is a passage to the limit in the two-body problem in Euclidean space $\mathbb{R}^3$. With the passage to the limit became the Kepler’s problem. There is an inertial system of reference, associated with the center of mass. The center of mass coincides with a body with mass which has increased. Let us consider the similar passage to the limit on $S^2$. In the case of $S^2$ with passage to the limit the heavy body moves freely (along geodesic), and the light particle moves in the field of massive one. Thus, because
of absence of the center of mass and the inertial system of reference the problem of two bodies on \( S^2 \) generally can not be reduced to the Kepler’s problem.

Let us consider a system of reference where the massive body doesn’t move. Then stationary configurations are critical points of effective potential

\[
U_{\text{eff}}(q) = -\frac{1}{2}(\omega \times q)^2 + U(q) = -\frac{\omega^2 R^2}{2}(\cos^2 \theta + \sin^2 \theta \cos^2 \varphi) - \gamma \cot \theta ,
\]

where

\[
\omega = (0, \omega, 0), \quad q = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta),
\]

and the massive body is situated on the axis \( Oz \). The stationary points of the potential (1.5) are given by

\[
\varphi = 0, \quad \omega^2 = \frac{2 \gamma}{R^2 \sin^2 \alpha \sin 2\alpha},
\]

where \( \alpha = |\theta| \) is a distance between bodies. From the equations (1.6) it follows, that all solutions are located in a region \( \alpha < \pi/2 \). So all configurations have the second type, and are obtained from (1.2) by passage to the limit when one of masses vanishes.

### 1.4. Stability of particular solutions of two-body problem

Let us consider stability of particular solutions of two-body problem in linear approximation. in the frame rotated with angular velocity \( \omega \) Hamiltonian, in spherical coordinates is written as:

\[
H = \sum_{i=1}^{2} \frac{p_i^2}{2m_i R^2} + \frac{p_{\varphi_i}^2}{2m_i R^2 \sin^2(\theta_i)} - p_{\varphi_i} \omega + U .
\]

Let us introduce variations of coordinates and momenta

\[
\delta \theta_i = \theta_i^0 + \delta \theta_i, \quad \delta \varphi_i = \varphi_i^0 + \delta \varphi_i,
\]

\[
\delta p_{\theta_i} = \delta p_{\varphi_i} \omega, \quad \delta p_{\varphi_i} = \delta p_{\varphi_i} + \delta p_{\varphi_i} .
\]

Substituting (1.8) to equations of motion for Hamiltonian (1.7) one can obtain the equations in variations of Poincaré

\[
\delta \dot{\theta}_i = \frac{\delta p_{\theta_i}}{m_i R^2},
\]

\[
\delta \dot{\varphi}_i = \frac{\delta p_{\varphi_i}}{m_i R^2 \sin^2 \theta_i} - \frac{2 p_{\varphi_i} \cos \theta_i}{m_i R^2 \sin^3 \theta_i} \delta \theta_i ,
\]

\[
\delta \dot{p}_{\theta_i} = \frac{2 p_{\varphi_i} \cos \theta_i}{m_i R^2 \sin^3 \theta_i} \delta p_{\varphi_i} - \frac{p_{\varphi_i}^2}{m_i R^2 \sin^4 \theta_i} (1 + 2 \cos^2 \theta_i) \delta \theta_i + \frac{\gamma m_1 m_2}{\sin^3 \alpha} (2 \delta_{ij} - 1) \delta \varphi_j ,
\]

\[
\delta \dot{p}_{\varphi_i} = - \frac{\gamma m_1 m_2}{\sin \theta_i \sin \theta_j} (2 \delta_{ij} - 1) \delta \varphi_j .
\]
Let us consider for particular solutions the stability depending on parameters \( \left( \frac{m_2}{m_1}, \alpha \right) \). The particular solutions of two-body problem for reduced system with Hamiltonian (1.7) are equilibrium points. So, the equations (1.9) will be linear differential equations with constant coefficients. The secular equation of this system is the equation of the fourth order with respect to \( x = \lambda^2 \). To make the particular solution stable in linear approximation it is necessary, that all \( \lambda \) of secular equations were purely imaginary, or, in other form, all \( x = \lambda^2 \) were real and negative. Two solutions \( \lambda \) of the secular equation of the system (1.9) are equal to zero. Eliminating these solution we shall receive a cubic equation on \( x = \lambda^2 \), which apparently has an analytical solution.

The regions on \( \left( \frac{m_2}{m_1}, \alpha \right) \) — plane with different types of solutions of the secular equation are given in Fig. 2. In region III the system has at least one eigenvalue with a positive real part, and is unstable already in linear approximation. In regions I and II all eigenvalues \( \lambda \) are purely imaginary. However, as it was shown above, two eigenvalues are equal to zero. The consederation of the linear approximation is not sufficient to determine the stability of the particular solution.

Now let us consider stability of particular solutions of the restricted two-body problem. The Hill zones for these problem are represented in Fig. 3. The points 1 and 2 are saddles of potential energy \( U_{eff} \) (1.5), and points 3, 4 are its maximums. The points 1, 3 and 2, 4 are merged pairwise in points 5 (\( \varphi = \pi, \theta = \frac{\pi}{6} \)) and 6 (\( \varphi = \pi, \theta = \frac{5\pi}{6} \)) when the curvature is increasing (at given \( \omega \)). The restricted two-body problem corresponds to the axis \( \frac{m_2}{m_1} = 0 \) in Figure 2. For \( \alpha \in [\pi/2, 2\pi/3] \) particular solution of the restricted two-body problem corresponds to maximum of effective potential (though there are no exponentially increasing solutions in linear approximation in this case), and for \( \alpha \in [2\pi/3, \pi] \) to a saddle point. Thus all stationary configurations in the restricted two-body problem are unstable.

The linear approximation can not give the complete answer to the question of stability of particular solutions of two-body problem. But with its help we can conclude, that in regions I and II only non exponential instability can exists, and therefore it is possible to consider restricted three-body problem.
2. Restricted three-body problem

As was shown above, in a constant curvature space there is a particular solution of two-body problem, when the distance between bodies is constant. So we can consider a restricted three-body problem in a curved space. The third particle (of zero mass) is added to the given system of two bodies, and this particle has no influence on motion of “heavy” bodies. As before, we shall examine the problem on invariant manifolds $S^2$ ($L^2$) (analog of a plane circular restricted three-body problem).

Consider a system, associated with two “heavy” bodies and rotating with an angular velocity $\omega$. The Lagrangian of a “light” particle in this system is written as:

$$ L = \frac{1}{2} (q + (\omega \times q), \dot{q} + (\omega \times \dot{q})) - U, \quad (2.1) $$

where $q$ are the cartesian coordinates of particle, $U = -\gamma m_1 \cot \alpha_1 - \gamma m_2 \cot \alpha_2$, $\alpha_1$ and $\alpha_2$ are angles between positions of vectors of the particle and “heavy” bodies (for $L^2 U = -\gamma m_1 \cot \alpha_1 - \gamma m_2 \cot \alpha_2$, $\alpha_1$ and $\alpha_2$ can be found from equations $\cosh \alpha_i = \langle q_i, q \rangle$). Thus we study the problem of the motion of a particle in the field of two Newtonian centres under gyroscopic forces.

The points of libration can be found as critical points of a reduced potential:

$$ U_* = U - \frac{1}{2} (\omega \times q, \omega \times q). \quad (2.2) $$

Let axis $Oz$ be directed along the angular velocity $\omega$, and fixed bodies be located in points

- $q_i (\sin \theta_i, 0, \sin \theta_i)$ on sphere $S^2$, and
- $q_i (\sinh \theta_i, 0, \cosh \theta_i)$ in space $L^2$.

In spherical (pseudospherical) coordinates the expression (2.2) for the reduced potential takes the form:

$$ U_* = -\gamma \sum_i \frac{m_i (\cos \theta_i \cos \theta + \sin \theta_i \sin \theta \cos \varphi)}{(1 - (\cos \theta_i \cos \theta + \sin \theta_i \sin \theta \cos \varphi)^2)^{1/2}} - \frac{1}{2} R^2 \omega^2 \sin^2 \theta \quad (2.3) $$

on sphere $S^2$, and

$$ U_* = -\gamma \sum_i \frac{m_i (\cosh \theta_i \cosh \theta + \sinh \theta_i \sinh \theta \cos \varphi)}{((\cosh \theta_i \cosh \theta + \sinh \theta_i \sinh \theta \cos \varphi)^2 - 1)^{1/2}} - \frac{1}{2} R^2 \omega^2 \sin^2 \theta \quad (2.4) $$

in space $L^2$.

By analogy with a plane space, it is possible to divide critical points of functions (2.3), (2.4) into two types:

a) **Collinear critical points** — generalization of Euler points of librations. They are situated in the plane of two fixed bodies and vector $\omega$ (futher $(m - \omega)$-plane).

b) **Noncollinear critical points** — generalization of Lagrangian points of libration, which, in case of a plane, are located on equal distances from attractive centres.

2.1. Collinear points of libration

For collinear points of libration $\varphi = 0$. From the symmetry of the problem we obtain that $\frac{\partial U_*}{\partial \varphi} = 0$ in the $(m - \omega)$-plane. Thus, we can to study critical points of a function

$$ f = U_* |_{\varphi=0} = -\sum_i \gamma m_i \cot \alpha_i | \theta - \theta_i | - \frac{1}{2} R^2 \omega^2 \sin^2 \theta. $$
These critical points correspond to the solutions of the following equation

$$\frac{1}{2} R^2 \omega^2 \sin \theta \cos \theta = \gamma \sum_i m_i \frac{\sin(\theta - \theta_i)}{\sin^3(\theta - \theta_i)},$$

where $\theta \in (-\pi, \pi)$.

The equation (2.5) can have 2, 4 or 6 solutions (depending on parameters). Their approximate position on a sphere is shown in Fig. 4. Scaling we choose the length of the arc between "heavy" bodies equal to the unity. So $R = \frac{1}{\alpha}$ ($\alpha$ is the angle between "heavy" bodies). Thus parameters vary within a rectangle $\{\frac{m_2}{m_1} \in [0, 1], \alpha \in [0, \pi]\}$. Let us consider separately cases $\alpha < \pi/2$ and $\alpha > \pi/2$, with different types of motion (see Section 1.2).

a) $\alpha < \pi/2$. When $\alpha$ is small there are six points of libration: $L_1, L'_1, L_2, L'_2, L_3, L'_3$. The points $L_1, L_2$ and $L_3$ tend to zero $\sim \alpha$ and transfer to Euler point of libration on a plane when $\alpha$ is vanishing ($R \to \infty$). At the same time the points $L'_2$ and $L'_3$ tend to the equator $\sim \pi/2 - \alpha^2$, and the point $L'_1$ moves to the south pole $\sim \alpha$.

The dependence of the position of libration points on $\alpha$ is given in Fig. 5 at $\frac{m_2}{m_1} = 0.2$. From this figure we can see that, six points of libration exist in interval of $\alpha$ from 0 up to $\alpha^*_1$. At $\alpha = \alpha^*_1$ the points $L_2$ and $L'_2$ are merged with each other and disappear. Thus there are four points of libration in interval $\alpha \in (\alpha^*_1, \alpha^*_2)$. The second pair of libration points ($L_3$ and$L'_3$) disappear at $\alpha = \alpha^*_2$ and only two Euler point of libration exist for $\alpha \in (\alpha^*_2, \alpha^*_3)$. $\alpha^*_3$ the pair of points $L_3$, $L'_3$ is born again at $\alpha = \alpha^*_3$ and then there are four points of libration till $\alpha = \pi/2$. The dependence of critical points $\alpha^*_1$, $\alpha^*_2$ and $\alpha^*_3$ on the ratio of masses $\frac{m_2}{m_1}$ is shown in Fig. 6. There are six, four, three and four libration points in regions I, II, III and IV accordingly.

b) $\alpha > \pi/2$. The dependence of the position of libration points on $\alpha$ in interval $(\pi/2, \pi)$ is shown in Fig. 7. When $\alpha$ is more then $\pi/2$, but less than some critical value $\alpha^*_4$ there are two central points of libration (on the north and south poles) and a pair of points near to the light body. The points $L_2$ and $L'_2$ are merged with each other and disappear at $\alpha = \alpha^*_4$. One more pair of libration points appears at $\alpha = \alpha^*_5$. The dependence of critical points $\alpha^*_4$ and $\alpha^*_5$ on the ratio of masses $\frac{m_2}{m_1}$ is shown in Fig. 8. From this figure it follows that the value $\alpha^*_4$ can be both less and more than $\alpha^*_5$ (in
dependence on the mass ratio). The curves $\alpha_1^*(\frac{m_2}{m_1})$ and $\alpha_2^*(\frac{m_2}{m_1})$ in this figure restrict regions of equal number of libration points. There are four points of libration in regions II, IV, and there are two and six ones in regions I and III accordingly.

2.2. Noncollinear libration points

a) $\alpha < \pi/2$. Noncollinear points of libration are defined by following system of equations:

$$\omega^2 R^2 \cos \theta = \gamma \sum_i m_i \frac{\cos \theta_i}{\sin^3 \alpha_i},$$

$$\sum_i m_i \frac{\sin \theta_i}{\sin^3 \alpha_i} = 0,$$

$$\cos \theta = \frac{\cos \alpha_2 \sin \theta_1 - \cos \alpha_1 \sin \theta_2}{\sin \alpha},$$

(2.6)

where $\alpha_i$ is the angle between the light particle and $i$-th body. At $m_1 = m_2$ the system (2.6) has solution for which $\alpha_1 = \alpha_2$. Opposite to the plane case, the relation $\alpha_1 = \alpha_2$ is not fulfilled at $m_1 \neq m_2$. The right side of the first equation of the system (2.6) is always positive. So, all Lagrangian points are always situated in the upper hemisphere. From the symmetry of the problem we can obtain that each point $L$ has a pair point $L'$ symmetric to $L$ relatively to ($m - \omega$)-plane. Therefore we can examine the problem only in region ($\varphi \in [0, \pi]; \theta \in [0, \pi/2]$).

The scan of a sphere with curves, on which libration points move with increasing of $\alpha$ is shown in Fig. 9 at $\frac{m_2}{m_1} = 0.6$. The points with primes but with identical numbers correspond to different points of libration at the same value of $\alpha$.

There is one point of libration close to north pole at small values of $\alpha$. Two Lagrangian points of libration are born in the point 2 when $\alpha = \alpha_1^*$. The point 3' at once moves to the first meridian with increasing of $\alpha$. And this point is joined with Euler point of libration at $\alpha = \alpha_2^*$. This merge changes the character of Euler point from the saddle of
a function $U$ to its minimum. With decreasing of the ratio of masses $\frac{m_2}{m_1}$ the point 2 tends to the first meridian and disappears at some critical value of $\frac{m_2}{m_1}$. When the point 2 has disappeared the Lagrangian point of libration is born directly from the Euler one. Then two remained Lagrangian points of libration move along the curve 1 and at $\alpha = \alpha_3^{**}$ join with each other and disappear.

The regions of equal numbers of libration points are shown in Fig. 10. The region IV has not any Lagrangian points of libration. There are 1, 3 and 2 Lagrangian points in regions I, II and III accordingly.

b) $\alpha > \pi/2$. Curve II in Fig. 9 corresponds to the values $\alpha > \pi/2$. A pair of Lagrangian points of libration is born at $\alpha = \alpha_i^{**}$. One of these points tends to the first meridian and joins with the central Euler point\(^1\) when $\alpha$ is increasing. At the same time the other point of libration moves to the equator. Thus, there are no libration points in region V (Fig. 10), two and one points in regions VI and VII accordingly.

For the Lobachevski plane $L^2$ in the equations (2.5) and (2.6) we shall change trigonometrical functions to hyperbolic ones. The analysis of the obtained equations shows, that three Euler and two Lagrangian points of libration exist at all values of parameters. And they turn into classical points of libration at $R \to \infty$. The curvature of space in this case gives only the changing of the position of libration points relatively to their position on a plane. The qualitative change of the picture in the case of $S^2$ arises from the superposition of compactness of space and its curvature.

2.3. Lagrangian points of libration in case of equal masses

Let us consider a case of equal masses of "heavy" bodies separately. In this case, because of additional symmetry, the analysis of behaviour of libration points can be done.

1. Case of $S^2$.

Letting $m_1 = m_2$ in the system (2.6) we get $|\theta_1| = |\theta_2|$ and $\alpha_1 = \alpha_2 = \alpha_0$. Thus the system (2.6) is reduced to the equation of one variable

$$\cos \alpha_0 \sin^3 \alpha_0 = C,$$  \hspace{1cm} (2.7)

where $C = \sin^3 \alpha \cos^2 \frac{\alpha}{2}$ is a constant depending on $\alpha$. The function in the left side of the equation (2.7) has maximum at $\alpha_0 = \pi/3$ with value $C^* = \frac{3\sqrt{3}}{16}$. Thus there are three opportunities:

1. $C(\alpha) > C^*$ — there are no Lagrangian points;
2. $C(\alpha) = C^*$ — there is one Lagrangian point;

\(^1\)Above we have considered only one hemisphere, therefore in fact there is a merge not of two, but of three points: two Lagrangian (by one in each hemisphere) and one Euler.
3. \( C(\alpha) < C^* \) — there are two Lagrangian points (as above only one hemisphere is considered). Let us rewrite (2.7) as

\[
\cos \alpha_0 \sin^3 \alpha_0 = \sin^3 \alpha \cos^2 \frac{\alpha}{2}.
\]

(2.8)

For sufficiently small \( \alpha \) the third condition is fulfilled, and there are two Lagrangian points:

\[
\alpha_0 \to \alpha,
\]

\[
\alpha_0 \to \pi/2.
\]

Thus one point in the case of limit \( R \to \infty \) becomes a classic Lagrangian point, and the second one tends to the equator. The dependence of position of libration points on \( \alpha \) is shown in Fig. 11.

Solving the equation

\[
\sin^3 \alpha \cos^2 \frac{\alpha}{2} = C^*
\]

(2.9)

we obtain two critical values \( \alpha_1^* \) and \( \alpha_2^* \). At \( \alpha = \alpha_1^* = 0.81487 \ldots \) the points of libration merge with each other and disappear in the point \( (\theta_1^* = \arccos \left( \frac{\cos \alpha_0}{\cos \alpha_1^*/2} \right) = 0.9949 \approx 57^\circ, \quad \varphi_1^* = \frac{\pi}{2} \)\). A new pair of libration points is born in the point \( (\theta_2^* = \arccos \left( \frac{\cos \alpha_0}{\cos \alpha_2^*/2} \right) = 0.5945 \approx 34^\circ, \quad \varphi_2^* = \frac{\pi}{2} \)\) at \( \alpha = \alpha_2^* = 1.8457 \ldots \). As it is shown in Fig. 11 there are no Lagrangian points of libration in interval \( \alpha \in (\alpha_1^*, \alpha_2^*) \). After the birth of a pair of libration points at \( \alpha = \alpha_2^* \) one of these points moves to the north pole with increase of \( \alpha \) and merges with Euler point of libration at \( \alpha = \alpha_3^* = \arccos \left( \frac{1}{\sqrt{2}} \right) = 1.8680 \ldots \). The equation (2.8) has two solutions at \( \alpha > \alpha_3^* \). However the existence condition of a point \( (\alpha_1, \alpha_2) \) on a sphere:

\[
|\alpha_2 - \alpha| < \alpha_1 < |\arccos(\cos(\alpha_2 + \alpha))|
\]

is not fulfilled for one of the solutions. The other libration point moves to the equator as \( \alpha \) increases up to \( \pi \).

2. **Case of \( L^2 \).**

For space \( L^2 \) in case of equal masses the analog of the system (2.6) is reduced to the equation similar to (2.7):

\[
\cosh \alpha_0 \sinh^3 \alpha_0 = C,
\]

(2.10)

where \( C = \sinh^3 \alpha \cosh^2 \frac{\alpha}{2} \). Differently from the spherical case the equation (2.10) has the only root for any \( \alpha \), and this solution \( \alpha_0(\alpha) \) is always less than \( \alpha \). The value of \( \alpha_0 \) tends to \( \alpha \) at small \( \alpha \), and \( \alpha_0 \to \alpha - \frac{1}{4} \ln 2 \) at large \( \alpha \). The dependence \((\alpha - \alpha_0)\) on \( \alpha \) is shown in Fig. 12.
2.4. Lagrangian points of libration in case of almost equal masses

Let us examine asymptotical behavior of libration points in case of almost equal masses of “heavy” bodies. Let \( m_1 = m, m_2 = m + \delta m \).

1. Case of \( S^2 \).

As was shown in Section 1.2 the shift of equilibrium positions of two bodies of almost equal masses is given by

\[
\begin{align*}
\theta_1 &= \frac{\alpha}{2} + \delta \theta, \\
\theta_2 &= -\frac{\alpha}{2} + \delta \theta, \\
\delta \theta &= \frac{\tan \alpha \delta m}{m}.
\end{align*}
\tag{2.11}
\]

Let \( \alpha_i = \alpha_0 + \delta \alpha_i \) (here \( \alpha_0 \) is taken from Section 2.3). Substituting this equality and equations (2.11) into the system (2.6), we obtain the first order in \( \frac{\delta m}{m} \)

\[
\begin{align*}
\alpha_1 &= \alpha_0 + \delta \alpha_0, \\
\alpha_2 &= \alpha_0 - \delta \alpha_0, \\
\delta \alpha_0 &= \frac{\sin^2 \frac{\alpha}{2} \tan \alpha_0}{6 \cos \alpha} \frac{\delta m}{m}.
\end{align*}
\tag{2.12}
\]

It is obvious that \( \delta \alpha_0 > 0 \) for \( \alpha < \pi/2 \), for any \( \alpha_0 (\alpha_0 < \pi/2 \text{ at equal masses, see Section 2.3}) \). Therefore the Lagrangian points move to the heavy body as its mass increases.

If \( \alpha > \pi/2 \), then \( \delta \alpha_0 < 0 \) and Lagrangian points move to the light body.

Let us consider variation of a curve, along which libration points move, and dependence of this variation on the ratio of masses \( (\delta \alpha_0 \text{ does not give complete picture, because bodies move with changing of masses}) \). Using coordinate transformation from \( (\alpha_1, \alpha_2) \) to spherical coordinates \( (\theta, \varphi) \) we obtain

\[
\begin{align*}
\cos \theta &= \frac{\cos \alpha_1 \sin \theta_1 - \cos \alpha_1 \sin \theta_2}{\sin \alpha}, \\
\sin \theta \cos \varphi &= \frac{\cos \alpha_1 \cos \theta_2 - \cos \alpha_2 \cos \theta_1}{\sin \alpha}.
\end{align*}
\tag{2.13}
\]

After substitution (1.4) and (2.12) into (2.13) we have in the first order in \( \frac{\delta m}{m} \)

\[
\begin{align*}
\delta \theta &= 0, \\
\delta \varphi &= \frac{\cos \alpha_0 \sin \alpha}{6 \cos \alpha \sin \theta_0} (\tan^2 \alpha_0 - 3) \frac{\delta m}{m}.
\end{align*}
\tag{2.14}
\]

If \( \alpha_0 < \pi/3 (\theta < \theta^*_1) \) and \( \alpha < \pi/2 \) then \( \delta \varphi < 0 \) and the curve moves to the light body. If \( \alpha_0 > \pi/3 (\theta > \theta^*_1) \), \( \alpha < \pi/2 \) then the curve moves to the heavy body. If \( \alpha > \pi/2 \), the curve moves in reverse directions (because of the replacement \( \theta^*_1 \) by \( \theta^*_2 \)). That is in the complete correspondence with results obtained above (see Fig. 9).

2. Case of \( L^2 \).

Let us make similar decomposition on \( \frac{\delta m}{m} \) for \( L^2 \). As a result we obtain

\[
\begin{align*}
\alpha_1 &= \alpha_0 + \delta \alpha_0, \\
\alpha_2 &= \alpha_0 - \delta \alpha_0, \\
\delta \alpha_0 &= -\frac{\sinh^2 \frac{\alpha}{2} \tanh \alpha_0}{6 \cosh \alpha} \frac{\delta m}{m}.
\end{align*}
\tag{2.15}
\]
In this case $\delta \alpha_0 < 0$ for any parameters. So points of libration always move to the light body. Values $\delta \theta$ and $\delta \varphi$ are rewritten as

$$
\delta \theta = 0,
$$

$$
\delta \varphi = \frac{\cosh \alpha_0 \sinh \frac{\varphi}{2}}{6 \cosh \alpha \sinh \theta_0} (\tan^2 \alpha_0 + 3) \frac{\delta m}{m}.
$$

From (2.16) it is follows that $\delta \varphi$ is always positive, i.e. the points of libration on $(\theta, \varphi)$-plane move to the heavy body.

2.5. Hill zones

Let us consider now Hill zones for restricted three-body problem on a sphere $S^2$. The complete bifurcation diagram for libration points on $S^2$ is shown in Fig. 13. The numbers of collinear and noncollinear libration points in each region on the diagram 13 are given in the Table 1. Different types of Hill zones correspond to regions on the diagram 13. Here we shall present Hill zones only for some regions on the diagram 13, which are more interesting as cases of existence of stable libration points. The Hill zones on $S^2$ are represented in Fig. 14, 15 and 16 for values of parameters from regions I, III, IV in Fig. 13 accordingly. The small-sized details of the Hill zones are shown on separate figures. The points 1–5 in all figures are the generalization of classic points of Lagrange and Euler. Points 6, 7 and 10 are new collinear points, points 8 and 9 are new Lagrangian points. In region I (Fig. 13) the point 6 is a minimum of potential energy, point 4 and 5 are its maximums, and all the rest points are saddles. Two pairs of Lagrangian points of libration are born with crossing of the boundary between regions I and II. It doesn’t influence on stability of the rest points. Crossing of the boundary II–III one of born pairs of Lagrangian points merges with Euler point and changes its type to a minimum of the effective potential. Thus there are two stable points 6 and 7 in region III. Further, crossing the boundary III–IV Euler points 2 and 6 merge with each other and disappear. Thus there is only one stable Euler point of libration in region IV.

Thus, new Euler libration points appear with the presence of curvature. Some of these libration points are stable at certain values of parameters, in particular in regions I, III and IV (Fig. 13).

![Fig. 13.](image)

The autor is grateful to A. V. Borisov and I. S. Mamaev for the setting up of the problem and useful discussions.
Table 1. Quantity of libration points in regions in Fig. 13.

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<th>Noncollinear</th>
<th>N of region</th>
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<th>Noncollinear</th>
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References


A. A. KILIN

ТОЧКИ ЛИБРАЦИИ В ПРОСТРАНСТВАХ $S^2$ И $L^2$

Поступила в редакцию 14 сентября 1998 г.

Рассматриваются задача двух тел и ограниченная задача трех тел в пространствах $S^2$ и $L^2$. Показано отсутствие экспоненциальной неустойчивости частных решений задачи двух тел, соответствующих круговому движению на плоскости. Найдены новые точки либрации, исследуется зависимость их положения от параметров системы, построены области существования точек либрации в пространстве параметров. На основе исследования областей Хила производится качественная оценка устойчивости точек либрации.