How to Control Chaplygin’s Sphere Using Rotors

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Abstract—In the paper we study the control of a balanced dynamically non-symmetric sphere with rotors. The no-slip condition at the point of contact is assumed. The algebraic controllability is shown and the control inputs that steer the ball along a given trajectory on the plane are found. For some simple trajectories explicit tracking algorithms are proposed.

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1. INTRODUCTION

The paper addresses the controllability of a dynamically non-symmetric balanced sphere moving over a plane. Internal rotors are used as the mechanism for control. The velocity of the point of contact is assumed to be zero (i.e., there is no slipping). Thus, we have a mechanical system with a non-holonomic constraint.

Over the past few decades dynamics and controllability of such systems have been intensively studied (see, for example, [5, 17, 18] and the references therein; a so-called non-holonomic manipulator is discussed in [20, 22]). Nowadays, the controllability of a sphere moving over various sorts of surfaces is widely discussed in the literature primarily in connection with the development of novel means of locomotion. Indeed, spherically-shaped robots seem to have a few advantages over conventional wheeled robots. The idea of spherical vehicle traces back to the second half of the XIX century. The advances in electronics and mobile robotics, which are mostly due to the space programs (especially the design of Earth and extraterrestrial rovers [50]), quickened interest in such vehicles at the end of the XX century. Extensive historical reviews on the subject can be found in [15, 23, 48].

There are a variety of engineering solutions to control a spherical robot, three of which (widely discussed in the literature) are as follows: 1) (the most popular one) control via changing the position of the center of mass using various internal actuators, such as sliders, pendulum-type mechanisms, small carts rolling inside the sphere, etc. [3, 14, 15, 19, 33, 37, 44, 49, 51], 2) control via internal rotors which are used to alter the angular momentum of the system; i.e., the sphere is controlled through the gyrostatic effect [28, 29], and 3) (now gaining popularity) control via shape deformation [47].

In this paper we consider a dynamically non-symmetric sphere with three balanced internal rotors. The center of mass of the sphere+rotors system is located at the geometric center of the sphere. It is not an accident choice because in this case we obtain an integrable problem (posed and solved by S. A. Chaplygin in 1903 [11]), provided, of course, the rotors do not spin; moreover, Chaplygin found a few intriguing geometric features of the trajectories of such a sphere. Some fresh results concerned with the behavior of Chaplygin’s sphere can be found in [31], where a
bifurcation analysis and computer simulations are carried out and some sample trajectories of the point of contact are presented. The case where the rotors rotate at a non-zero speed was explored in detail by Bobylev [1] and Joukovski [6] but they assumed that the body was axisymmetric and the gyrostatic momentum of the rotors was aligned with an axis of dynamical symmetry.

The integrability of the system of dynamically non-symmetric sphere with an arbitrary constant gyrostatic momentum was established by A. P. Markeev [7] in 1986. A computer analysis of the trajectories of the point of contact was partly performed in [31], and the paper [9] contains a topological analysis of stability of partial motions of a sphere. The approach used in [9] is based on the methods of analysis of integrable systems developed in [2] which, in their turn, rest on a concept (introduced by the authors) of bifurcation complex. Unfortunately, the fact that the system of Chaplygin’s sphere and a gyrostat is integrable (which was rigorously established in [7]) is practically unknown among the robotic community. But it should be noted that this property of integrability is intimately connected with the existence of the vector of generalized angular momentum which is fixed in space. In its turn this vector (at least theoretically) seriously facilitates the use of the rotors as control mechanism. In particular, on the common level surface of these three integrals of motion one gets a system of the first order ordinary differential equations whose right sides depend linearly on the control inputs which allows one to develop explicitly a motion planning algorithm. Moreover, the inputs can be explicitly found as algebraic functions of the coordinates of the point of contact; this simplifies the whole matter a lot and shows promises that for this system many of the standard control tasks can be solved completely. This paper addresses one such a problem; namely, we study the controlled dynamics of the system on the zero level of the momentum integral, i.e., the controlled system is driftless (thus, the sphere is assumed to start from rest). In this case we propose explicit tracking algorithms for some simple trajectories. In subsequent publications we will consider some obvious extensions of this problem, e.g. controlled motion of a sphere on an inclined plane or curved surfaces, and other standard issues from the control theory, such as stabilization, optimal control, feedback control, etc.

In conclusion, we mention some contributions that demonstrate various approaches to this problem. First of all, control of non-holonomic systems is considered in [13, 36, 38, 41]. There is quite a number of more theoretically oriented researches based on modern geometric dynamics, reduction and sub-Riemannian geometry [30, 45, 46]); such results are hardly relevant to real applications and usually poorly understood by engineers. On the other hand, there exist what may be called “purely engineering” (or “totally practical”) works (see, for example, [28, 29]) where awkward, inconvenient and even erroneous equations of motion (if any at all) are used. Noteworthy are the publications [39, 40] in which various control strategies and motion-planning algorithms are discussed. So this paper is intended to be a compromise as it may be of some interest from the point of view of the classical mechanics and control theory (because the problem of control for a particular mechanical system is explicitly studied). On the other hand, we tried to make the exposition (and the equations of motion) maximally accessible to engineers so as for them to be capable of using this paper as a solid (maybe idealized) foundation on the basis of which a technically complicated robotic system (with all sorts of sensors, stabilizing systems and maybe elements of artificial intelligence) can be created.

It should be noted that the “classical” model of rolling motion which is adopted in the above mentioned publications (as well as in this paper) is hardly always relevant. Indeed, at the contact spot there is dissipation, which the model does not account for. As a result, for large times the system deviates from the “base” trajectory obtained in the context of the “classical” non-holonomic model. To correct the trajectory, it is customary to set up a feedback loop using sensors mounted on the sphere. The main advantage of the non-holonomic model as compared to other (usually more complicated) models that incorporate friction is that it allows for analytic expressions for the inputs.

In the control theory, there is another contact model, besides the “classical” one, which has received much recognition. The latter model assumes that at the point of contact not only sliding but also spinning about the vertical is forbidden (i.e., the projection of the body’s angular velocity onto the vertical axis must be zero) [12, 23, 27, 38]. Interestingly, in the realm of the control theory and the robotic science this model is commonly accepted, while its exploration by specialists in the non-holonomic dynamics has been started only recently [4, 32]. J. Koiller aptly dubbed the two models as marble-rolling (for the “classical” model) and rubber-rolling (for the no-spinning
model). It should be noted that unlike the “classical” case, mechanical systems subject to the no-spin condition are practically unexplored [4] and deserve a fresh look and more research. Moreover, in the future it seems reasonable to study (both theoretically and experimentally) not only non-holonomic models but also models that account for friction.

The paper is dedicated to the 70-th birthday of Prof. A. P. Markeev.

2. EQUATIONS OF MOTION AND THEIR FIRST INTEGRALS

Consider a sphere with three internal rotors. The sphere rolls on a plane without slipping (Fig. 1). We adopt the following assumptions:

- the center of mass of the sphere + rotors system is located at the geometric center of the sphere;
- all the rotors are identical, possess axial symmetry and their spinning directions are aligned with their axes of symmetry, meaning that the rotors cannot change the distribution of mass;
- the spinning directions of the rotors are non-coplanar, their angular velocities $\omega_k(t), k = 1, 2, 3$ are given functions of time.

Choose a body-fixed frame $Oe_1e_2e_3$ whose axes are aligned with the principal axes of inertia of the sphere+rotors system (Fig. 1). Let $V$ and $\Omega$ be the velocity of the sphere’s center and the angular velocity of the sphere. Hereinafter all vectors are assumed to be referred to this body-fixed frame (unless otherwise stated). The no-slip condition can be written as

$$ f = V + \Omega \times r = 0, \quad (2.1) $$

where $r = -R\gamma$ is the radius-vector from the sphere’s center to the point of contact, $R$ is the radius of the sphere and $\gamma$ is a unit vector normal to the plane (Fig. 1).

Let $O'xyz$ be a laboratory (fixed) frame of reference with the axis $O'z$ perpendicular to the plane. Denote by $r_{\ldots} = (x, y)$ the coordinates of the point of contact $C$ in this frame. Let $\alpha, \beta$ and $\gamma$ be the unit vectors along the axes of the laboratory frame and written in terms of their projections onto the axes of the body frame $e_1, e_2, e_3$. Then the orthogonal matrix

$$ Q = \begin{pmatrix}
\alpha_1 & t_1 & \gamma_1 \\
\alpha_2 & t_2 & \gamma_2 \\
\alpha_3 & t_3 & \gamma_3
\end{pmatrix} \in SO(3) \quad (2.2) $$

accounts for the body’s orientation and the pair $(r_{\ldots}, Q)$ uniquely defines the configuration of the system. Thus, the five-dimensional configuration space of the system is the product $\mathbb{R}^2 \times SO(3)$; equation (2.1) fixes a three-dimensional non-holonomic distribution in this space.
The equations of motion for our system, which is subject to the constraint (2.1), can be written in the form of the Ferrers equations with undetermined multipliers (also known as the Poincaré–Suslov–Kozlov–Fedorov equations)

\[
\left( \frac{\partial T}{\partial \dot{V}} \right)^{\cdot} + \Omega \times \frac{\partial T}{\partial \dot{V}} = \sum_{k=1}^{3} \lambda_k \frac{\partial f_k}{\partial \dot{V}}, \quad \left( \frac{\partial T}{\partial \Omega} \right)^{\cdot} + \Omega \times \frac{\partial T}{\partial \Omega} = \sum_{k=1}^{3} \lambda_k \frac{\partial f_k}{\partial \Omega}.
\]

Here \( \lambda_k \) are undetermined multipliers and \( T \) is the kinetic energy of sphere + rotors system.

Let \( m \) and \( I = \text{diag}(I_1, I_2, I_3) \) be the mass and the tensor of inertia of the sphere+rotors system. Denote by \( i \) the moment of inertia of a rotor about the spinning direction and let \( n_k, k = 1, 2, 3 \) be the unit vectors along these directions. In the body-fixed frame the components of the vectors \( n_k \) are constant. The kinetic energy is the sum of the kinetic energy of the sphere and that of the rotors:

\[
T_0 = \frac{1}{2} m_0 V^2 + \frac{1}{2} (\Omega, I_0 \Omega)
\]

and that of the rotors

\[
T_k = \frac{1}{2} m_k V^2 + \frac{1}{2} (\Omega + \omega_k n_k, I_k (\Omega + \omega_k n_k)), \quad k = 1, 2, 3.
\]

Here \( m_0, m_k, I_0, \) and \( I_k \) are the masses and the tensors of inertia relative to the frame \( O e_1 e_2 e_3 \) of the sphere and the rotors, respectively. Since the spinning directions are aligned with the eigenvectors of the tensor of inertia, i.e., \( I_k n_k = i n_k \), we have

\[
T = T_0 + \sum_k T_k = \frac{1}{2} m V^2 + \frac{1}{2} (\Omega, \Omega) + (\Omega, K(t)) + \frac{1}{2} \sum_k i \omega_k^2(t),
\]

where \( K(t) = \sum_k i \omega_k(t) n_k \) is the vector of gyrostatic momentum.

Using the equation of constraints (2.1) and the equations of motion (2.3), we find the multipliers to be

\[
\lambda = m r \times \dot{\Omega},
\]

where \( \lambda = (\lambda_1, \lambda_2, \lambda_3) \). Plugging this into (2.3) yields

\[
(\Omega + K)^{\cdot} + m r \times (\dot{\Omega} \times r) = (I \Omega \times K) \times \dot{\Omega}.
\]

The time evolution of the variables \((r, \omega, \Omega)\) is governed by the Poisson equation and the kinematic relations (which are a consequence of the no-slip condition), that is,

\[
\dot{\alpha} = \alpha \times \Omega, \quad \dot{\beta} = \beta \times \Omega, \quad \dot{\gamma} = \gamma \times \Omega,
\]

\[
\dot{x} = R(\beta, \Omega), \quad \dot{y} = -R(\alpha, \Omega).
\]

Equations (2.6) and (2.7) constitute a full system of governing equation for the sphere+rotors system.

In view of the orthogonality condition, the equations admit the following geometric integrals of motion:

\[
\alpha^2 = \beta^2 = \gamma^2 = 1, \quad (\alpha, \beta) = (\alpha, \gamma) = (\beta, \gamma) = 0.
\]

Define the vector of the angular momentum of the system about the point of contact as

\[
M = I \Omega + D \gamma \times (\Omega \times \gamma) + K(t).
\]

It can be shown that that it evolves with time as follows:

\[
\dot{M} = M \times \Omega.
\]

Therefore, \( M \) is a constant vector in the laboratory frame and thus its projections onto the axes of this frame are integrals of motion, that is,

\[
M_\alpha = (M, \alpha) = \text{const}, \quad M_\beta = (M, \beta) = \text{const}, \quad M_\gamma = (M, \gamma) = \text{const}.
\]
Remark. If the rotors spin at a constant speed, then the energy of the system
\[ E = \frac{1}{2}(M, \Omega) \]
is also a conserved quantity and therefore the system is integrable [7]. In the generic case (\( \omega \) depends on \( t \)) the energy is no longer a constant.

3. CONTROLLABILITY ON THE ZERO LEVEL OF MOMENTUM

Let us see how the sphere can be controlled by the vector of gyrostatic momentum \( \mathbf{K}(t) \). To do this we restrict our system to the common level of the integrals (2.10) and write the equations of motion in the form accepted in the control theory. Solving (2.9) for \( \Omega \) we get

\[
\begin{align*}
\dot{\alpha} &= \alpha \times (\Omega_0 + \Omega_c), & \dot{\beta} &= \beta \times (\Omega_0 + \Omega_c), & \dot{\gamma} &= \gamma \times (\Omega_0 + \Omega_c), \\
\dot{x} &= R(\beta, \Omega_0 + \Omega_c), & \dot{y} &= -R(\alpha, \Omega_0 + \Omega_c), \\
\Omega_0 &= S(M_\alpha \alpha + M_\beta \beta + M_\gamma \gamma), & \Omega_c &= -SK(t), \\
S &= A + (D^{-1} - (\gamma, A\gamma))^{-1}(A\gamma) \otimes (A\gamma), & A &= (I + DE)^{-1}.
\end{align*}
\]

Thus, using the momentum integral (2.10), we reduce the original equations in the phase space to the equations in the configuration space and thereby seriously simplify the subsequent analysis.

Remark. Strictly speaking, the inputs in this system are the voltages \( u_k(t) \), \( k = 1, 2, 3 \) on the electric motors that drive the rotors. In reality, however, it is customary to make use of a linear approximation for an electric motor’s torque [8]:

\[ i\dot{\omega}_k = c_u u_k(t) - c_\omega \omega_k, \quad k = 1, 2, 3, \]

where \( c_\omega \omega_k \) is the moment of the counter-electromotive force and the coefficients \( c_u \) and \( c_\omega \) depend on the parameters of the motor [8]. Moreover, the angular velocities \( \omega_k(t) \) are connected to the gyrostatic moment by (2.9); therefore, given a control law \( \mathbf{K}(t) \), the voltages can be found as follows

\[ \mathbf{u}(t) = c_u^{-1}N^{-1}\left(\mathbf{K}(t) + \frac{c_\omega}{i}\mathbf{K}(t)\right), \quad \mathbf{u}(t) = (u_1(t), u_2(t), u_3(t)). \]

Here the matrix \( N = ||n_{kl}|| \) is composed of the vectors \( n_k \). We assume that the three motors are identical.

For \( \Omega_c = 0 \) (or \( \Omega = \Omega_0 \)) the system turns into the celebrated Chaplygin’s sphere problem, which is considered in numerous publications (for example, see the references in [31]). In the rest of the paper we assume that \( \Omega_0 = 0 \), i.e., there is no drift. This is equivalent to the condition that all the integrals (2.10) are identically zero:

\[ M_\alpha = M_t = M_\gamma = 0. \]

Moreover, we will confine ourselves to the following problem.

**Suppose that at initial time \( t = 0 \) the sphere is at rest and the angular velocities of the rotors are zero. We seek to specify the inputs \( \mathbf{K}(t), t \in [0, T], T < \infty \) that steer the sphere into a prescribed position in time \( T \), more exactly, when \( t = T \) the sphere must stop at a specified point and have a specified orientation.**

(3.2)

In this section we will prove the controllability of our system using the Chow–Rashevsky theorem. Rashevsky [10] formulated the theorem as follows:

**Theorem 1.** If among the vector fields \( \mathbf{X}_1, \ldots, \mathbf{X}_m \) and the fields generated from them by the Lie-bracket operation one can pick \( n \) fields \( \mathbf{Y}_1, \ldots, \mathbf{Y}_n \) which are linearly independent at each point of a domain \( \mathcal{G} \) such that \( \dim \mathcal{G} = n \), then starting from any point in \( \mathcal{G} \) one can reach any other point, travelling along the integral curves of the fields \( \mathbf{X}_1, \ldots, \mathbf{X}_m \).
Suppose a \( m \times m \) matrix \( S \) is non-degenerate at each point \( z \in G \); then this theorem remains valid for new fields \( \tilde{X}_1, \ldots, \tilde{X}_m \) defined as

\[
X_k = \sum_{k'=1}^m S_{kk'}(z)\tilde{X}_{k'}, \quad k = 1, \ldots, m.
\]

Indeed, we have

\[
[X_l, X_k] = \sum_{c',k'} S_{ll'}S_{kk'}[\tilde{X}_{l'}(\tilde{X}_{k'}) + \sum_{k''} \left(\sum_{l''} (S_{ll''}\tilde{X}_{l''}(S_{k''}) - S_{k''}\tilde{X}_{l''}(S_{ll''}))\right)\tilde{X}_{k''}].
\]

Thus, we see that the spans of the two vector fields coincide and so do the two spans generated by their commutators. Here we use the fact that the tensor product \( S \otimes S \) is a non-degenerate operation, meaning that for two \( n \times n \) matrices \( A \) and \( B \) one has \( \det(A \otimes B) = (\det A)^n(\det B)^n \). The argument can be extended to the case of iterated brackets with arbitrarily many factors.

Despite the trivial algebraic form of (3.1), it will be more convenient to use local coordinates on the configuration space \( M = \mathbb{R}^2 \times SO(3) \). The orthogonal matrix (2.2) can be parameterized with the Euler angles:

\[
Q = \left(\begin{array}{ccc}
\cos \varphi \cos \psi - \cos \theta \sin \psi \sin \varphi & \cos \varphi \sin \psi + \cos \theta \cos \psi \sin \varphi & \sin \varphi \sin \theta \\
-\sin \varphi \cos \psi - \cos \theta \sin \psi \cos \varphi & -\sin \varphi \sin \psi + \cos \theta \cos \psi \cos \varphi & \cos \varphi \sin \theta \\
\sin \theta \sin \psi & -\sin \theta \cos \psi & \cos \theta
\end{array}\right).
\] (3.3)

Denote the local coordinates on \( M \) as \( z = (\theta, \varphi, \psi, x, y) \) and write equations (3.1) in these coordinates

\[
\dot{z} = \sum_{l=1}^3 K_l(t)X_l(z) = -\sum_{k=1}^3 \Omega_{ek}\tilde{X}_k(z),
\]

\[
\Omega_e = -SK.
\]

Using the matrix \( S(\gamma) = \|S_{lk}\| \) we can write

\[
X_l = \sum_{k=1}^3 S_{lk}\tilde{X}_k,
\]

where

\[
\tilde{X}_1 = \left(\begin{array}{c}
\cos \varphi, -\cot \theta \sin \varphi, \\frac{\sin \varphi}{\sin \theta},
\end{array}\right),
\]

\[
R(\cos \varphi \cos \psi + \cos \theta \sin \psi \cos \varphi, -R(\cos \varphi \cos \psi - \cos \theta \sin \varphi \sin \psi)
\]

\[
\tilde{X}_2 = \left(\begin{array}{c}
-\sin \varphi, -\cot \theta \cos \varphi, \\frac{\cos \varphi}{\sin \theta},
\end{array}\right),
\]

\[
-R(\sin \varphi \cos \psi - \cos \theta \cos \varphi \cos \psi, R(\sin \varphi \cos \psi + \cos \theta \cos \varphi \sin \psi)
\]

\[
\tilde{X}_3 = (0, 1, 0, -R \sin \theta \cos \psi, -R \sin \theta \sin \psi).
\]

The fields \( \tilde{X}_k(z) \) look comparatively simple, so the conditions of the Chow–Rashevsky theorem can be easily verified. The Lie brackets of the fields read

\[
Y_1 = [\tilde{X}_1, \tilde{X}_3] + \tilde{X}_2
\]

\[
= (0, 0, 0, R(\sin \varphi \sin \psi - \cos \theta \cos \varphi \cos \psi), -R(\sin \varphi \cos \psi + \cos \theta \cos \varphi \sin \psi)),
\]

\[
Y_2 = [\tilde{X}_2, \tilde{X}_3] - \tilde{X}_1
\]

\[
= (0, 0, 0, R(\cos \varphi \sin \psi + \cos \theta \sin \varphi \cos \psi), -R(\cos \varphi \cos \psi - \cos \theta \sin \varphi \sin \psi)).
\]
It is straightforward to show that the vector fields $\tilde{X}_1, \tilde{X}_2, \tilde{X}_3, Y_1, \text{ and } Y_2$ are linearly independent, and therefore

**Proposition 1.** Chaplygin’s sphere with three rotors whose axes are non-coplanar is controllable ($z$ being the controlled variables) on the zero level of the angular momentum calculated about the point of contact.

According to the Chow – Rashevsky theorem the path from an initial point $z_0$ to a final point $z_1$ can be composed of the integral curves of the fields $\tilde{X}_1, \tilde{X}_2, \text{ and } \tilde{X}_3$; in modern literature such an approach to the motion planning problem is loosely called *gait control* [35, 42].

**Remark.** In the case of three rotors, at each point in $M$ the span of the control vector fields $\tilde{X}_1, \tilde{X}_2, \text{ and } \tilde{X}_3$ coincides with the non-holonomic distribution given by the constraint equation (2.1).

If there are only two rotors, then the vector of gyrostatic momentum reads

$$K = i\omega_1 n_1 + i\omega_2 n_2,$$

where $n_1$ and $n_2$ are not parallel.

This means that the components of the vector $K$ cannot be chosen arbitrarily; the equations on the group (3.1) allow the following representation

$$\dot{z} = \sum K_i X_i = \omega_1 X_1' + \omega_2 X_2',\quad X_1' = n_{11} X_1 + n_{12} X_2 + n_{13} X_3,\quad X_2' = n_{21} X_1 + n_{22} X_2 + n_{23} X_3,$$

where $n_{kl}$ are the components of the vectors $n_k, k = 1, 2$. Nevertheless, one can straightforwardly verify that the span of the fields $X_1', X_2'$, and $[X_1', X_2']$ coincides with the span of $X_1, X_2,$ and $X_3$, hence the conditions of the Chow – Rashevsky theorem are met in this case as well.

**Proposition 2.** Chaplygin’s sphere with two rotors whose axes are not parallel is controllable on the zero level of the angular momentum calculated about the point of contact.

If the momentum (2.10) is different from zero, then a *non-zero drift term* ($\Omega_0 \neq 0$) occurs in equations (3.1); if in addition $\Omega_0 = 0$ (there are no other terms but drift) equations (3.1) become identical to the equations of the integrable problem on free motion of Chaplygin’s sphere. Controlled systems with drift are treated in [16] (see also [24]) where it is shown that for such a system to be controllable not only the vector fields and their iterated Lie brackets must be full rank, but also in the case of pure drift the phase space of the system must contain an everywhere dense subset each point of which is stable in the sense of Poisson (for brevity, say that the drift is Poisson stable). On the other hand, judging from various investigations of the dynamics of Chaplygin’s sphere [11, 25, 31] it seems reasonable to hypothesize that almost all trajectories of the point of contact are not bounded. So the subject requires a more profound examination.

**Remark.** Recall that for a system $\dot{z} = v(z)$ a point $z_0$ is stable in the sense of Poisson if for the solution $z(t)$ that emanates from $z_0$ (i.e., $z(0) = z_0$) there exist two sequences such that $t_k^+ \to +\infty$, $t_k^- \to -\infty$ and

$$\lim_{k \to \infty} z(t_k^+) = \lim_{k \to \infty} z(t_k^-) = z_0.$$

**Remark.** For completeness sake we present Chow’s formulation of the controllability theorem [21] (rewritten in the modern notation).

**Theorem 2.** Consider a system of vector fields $B_r = \{X_1, \ldots, X_r\}$ on a manifold $M$ ($\dim M = n$); the full extension, $B_s$, of the system consists of the fields $X_k$ and all their iterated Lie brackets. Suppose the extension $B_s$ is regular at a point $a$; then there exist $n-s$ integrals $\varphi_1(x), \ldots, \varphi_{n-s}(x)$ such that for each point on the integral manifold $\varphi_1(x) = \varphi_1(a), \ldots, \varphi_{n-s}(x) = \varphi_{n-s}(a)$ (which obviously contains $a$) there exists a path that connects this point and the point $a$. This path can be composed of the trajectories of the system $B_r$.

It should be noted that Chow himself did not use the Lie bracket operation; he built the extension $B_s$ via infinitesimal transformations of the phase flows of the system $B_r.$
4. EXPLICIT TRAJECTORY TRACKING

4.1. Geometric and Dynamical Features of Rolling Motion at the Zero Level of Momentum

In the previous section the controllability of Chaplygin’s sphere actuated by rotors has been established. This means that (theoretically) the sphere can be steered from an initial state \( z_0 = (\theta_0, \varphi_0, \psi_0, x_0, y_0) \) to a final state \( z_f = (\theta_f, \varphi_f, \psi_f, x_f, y_f) \). Now we will develop an explicit algorithm that steers the sphere along a given trajectory on the plane. Prior to this, recall a few simple geometric statements concerned with sliding-free motion of a sphere over a plane. In this section the angular momentum relative to the point of contact is assumed to be zero, that is,

\[
M = I\Omega + D\gamma \times (\Omega \times \gamma) + K(t) = 0. \tag{4.1}
\]

**Remark.** Obviously, for a solution to the tracking problem to exist the length of the curve \( r(s) \) on the plane must be greater or equal than the geodesic distance between the points \( r(s_0) \) and \( r(s_f) \) on the sphere.

First of all let us clarify the geometric meaning of the Euler angles in (3.3); the general transformation \( Q \) is the composition of three rotations (Fig. 2):

\[
Q = Q_3(\varphi)Q_1(\vartheta)Q_3(\psi),
\]

\[
Q_1(u) = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos u & \sin u \\
0 & -\sin u & \cos u
\end{pmatrix}, \quad Q_3(u) = \begin{pmatrix}
\cos u & \sin u & 0 \\
-\sin u & \cos u & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

and \( \gamma = (\sin \theta \cos \varphi', \sin \theta \sin \varphi', \cos \theta), \ \varphi' = \frac{\varpi}{2} - \varphi \).

Recall that the radius-vector from the center of the sphere \( O \) to the point of contact \( C \) satisfies the relation \( r = -R\gamma \) (see (2.1)). Hence, the angles \( (\theta, \varphi) \) can be considered as the spherical coordinates of the point of contact in the body-fixed frame \( Oe_1 e_2 e_3 \). Thus, the location and orientation of the sphere on the plane can be uniquely defined in the following way:

choose a point on the plane with coordinates \((x, y)\) and a point on the sphere with spherical coordinates \((\theta, \varphi)\) (as discussed above); bring these points into coincidence and rotate the sphere about the vertical until the angle between the line of nodes \( ON \) and the axis \( Ox \) becomes equal \( \psi \),

where the line of nodes \( ON \) is the intersection of the plane \( Oe_1 e_2 \) and the horizontal plane \( O\alpha\beta \).
Let us mention another simple geometric property of a sliding-free rolling motion.

**Proposition 3.** Suppose that in the course of rolling the point of contact C describes a smooth curve \( r_{-}(t) = (x(t), y(t)) \) on the plane and a smooth curve \( r(t) = -R\gamma(t) \) on the sphere, where \( t \) is time; then the velocities with which the point moves along the two curves coincide. In the body-fixed frame this can be written as

\[
\dot{x}\alpha + \dot{y}\beta = -R\gamma. \tag{4.3}
\]

The orientation \( Q \) of the sphere is unambiguously defined at all times, and the nutation angle \( \psi \) can be found from the equations

\[
\sin \psi = \frac{\dot{x}\dot{\theta} - \dot{y}\dot{\phi}\sin \theta}{R(\dot{\theta}^2 + \sin^2 \theta\dot{\phi}^2)}, \quad \cos \psi = -\frac{\dot{y}\dot{\theta} + \dot{x}\dot{\phi}\sin \theta}{R(\dot{\theta}^2 + \sin^2 \theta\dot{\phi}^2)}. \tag{4.4}
\]

**Proof.** Indeed, the velocity of the point of contact \( C \) coincides with the velocity of the center of the sphere \( O \), and in the body-fixed frame this velocity reads \( \dot{V} = \dot{x}\alpha + \dot{y}\beta \); by virtue of (2.7), we have \( \Omega \times r = -R\Omega \times \gamma = R\dot{\gamma} \). Substituting these relations into the equation of constraint (2.1) yields the desired relation (4.3). Multiplying both sides of (4.3) by \( Q \), which is given by (3.3), we get

\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} =
\begin{pmatrix}
\cos \psi & -\sin \psi \\
\sin \psi & \cos \psi
\end{pmatrix}
\begin{pmatrix}
-R\sin \theta\dot{\phi} \\
-R\dot{\theta}
\end{pmatrix}. \tag{4.5}
\]

Solving this system for \( \cos \psi \) and \( \sin \psi \) gives (4.4).

As a corollary we see that the curves traced out by the point of contact on the plane and the sphere have the same length. Hereafter we will parameterize both curves by the arc length parameter \( s \).

**Remark.** It follows from (4.5) that at the point of contact the tangent vector to the plane curve \( \tau_p = \frac{\partial r_{-}(s)}{\partial s} \) calculated in the laboratory frame of reference and the tangent vector to the spherical curve \( \tau_b = \frac{\partial r(s)}{\partial s} \) calculated in the body-fixed frame can be brought into coincidence by a rotation by the angle \( 2\pi + \psi \) about the vertical.

Using these properties, we now proceed with two quite “natural” propositions which facilitate the development of control strategies.

**Proposition 4.** Suppose that \( M = 0 \) and the gyrostatic momentum is

\[
K(t) = \mu(t)I\gamma_0, \tag{4.6}
\]

where \( \mu(t) \) is an arbitrary function and \( \gamma_0 \) is the unit vector normal to the plane at the point of contact \( (r_0 = -R\gamma_0) \); then the sphere rotates about the vertical, while its center is fixed.

**Proof.** By virtue of (2.7), for such a motion the following relations hold

\[
\Omega \times \gamma_0 = 0, \quad (\Omega, \alpha) = (\Omega, \beta) = 0.
\]

These equations are simultaneous and have a non-zero solution of the form

\[
\Omega = \mu\gamma_0,
\]

where \( \mu(t) \) is an arbitrary function. Substituting \( \Omega \) into (4.1), we get

\[
K(t) = \mu(t)I\gamma_0. \tag*{□}
\]

**Proposition 5.** Suppose that \( M = 0 \) and there are two smooth curves of equal length: 1) an in-plane curve \( r_{-}(s) = (x(s), y(s)) \) and 2) a spherical curves \( r(s) = -R\gamma(s) \); then the gyrostatic momentum can be defined uniquely as a smooth function of time, namely,

\[
K(t) = -\dot{s}(t) \left( (I + DE) \left( \frac{d\gamma}{ds} \times \gamma \right) + (k_p - k_b)I\gamma \right),
\]

\[
k_p = \frac{dx}{ds} \frac{d^2y}{ds^2} - \frac{dy}{ds} \frac{d^2x}{ds^2}, \quad k_b = R^2 \left( \frac{d\gamma}{ds} \times \frac{d^2\gamma}{ds^2} \right), \tag{4.7}
\]

such that the point of contact \( C \) moves along the two curves; here \( s(t) \) specifies the law of motion.
Proof. To find the vector of gyrostatic momentum, rewrite equations (4.1) in the form

\[-K(t) = (I + DE)\Omega - D\Omega \gamma,\]

where \(\Omega = (\Omega, \gamma)\) is the normal component of the angular velocity. Taking the vector product of the Poisson equation \(\dot{\gamma} = \gamma \times \Omega\) with \(\gamma\), we obtain

\[\Omega = \dot{\gamma} \times \gamma + \Omega \gamma.\]

Using these relations, represent the sought-for gyrostatic momentum in the form

\[K(t) = -(I + DE)(\dot{\gamma} \times \gamma) - \Omega \gamma.\]  

(4.8)

Since \(\gamma(s)\) is given, we immediately find \(\dot{\gamma} = s(t) \frac{dy}{ds}\) and thereby obtain the first term. To find \(\Omega \gamma\) we use the following statement [34].

Lemma 1. If a sphere rolls without slipping on a plane, then

\[k_p = k_b + \frac{\Omega}{s},\]  

(4.9)

where \(k_p\) is the curvature of the plane curve at the point of contact and \(k_b\) is the geodesic curvature of the spherical curve; \(k_b\) is equal to the curvature of the projection of this curve onto the plane at the point of contact.

Calculating these curvatures, we get

\[k_p = \frac{dx}{ds} \frac{d^2y}{ds^2} - \frac{dy}{ds} \frac{d^2x}{ds^2}, \quad k_b = R^2 \left( \gamma, \frac{d\gamma}{ds} \times \frac{d^2\gamma}{ds^2} \right).\]

Substituting these expressions into (4.9) and (4.8) yields the desired relation. \(\square\)

4.2. Steering the Sphere along a Reference Trajectory

Using the propositions from the previous subsection, we present an algorithm that explicitly solves the control task (3.2). Moreover, we will show how to steer the sphere to a final configuration in such a way that the point of contact follows a prescribed curve in the plane.

As before, let \(z_0 = (\theta_0, \varphi_0, \psi_0, x_0, y_0)\) and \(z_f = (\theta_f, \varphi_f, \psi_f, x_f, y_f)\) be initial and final configurations of the sphere.

1. Connect the points on the sphere with coordinates \((\theta_0, \varphi_0)\) and \((\theta_f, \varphi_f)\) with a curve \(r(s)\) whose length equals the length of the plane curve.

2. Rotate the sphere about the vertical (see (4.6)) so that the tangent to the plane curve coincides with the tangent to the spherical curve. To do this, solve (4.4) for \(\psi_1\):

\[\psi_1 = \arctg \left( \frac{\sin \theta \frac{dy}{ds} \frac{d\varphi}{ds} - \frac{dx}{ds} \frac{d\theta}{ds}}{\sin \theta \frac{dx}{ds} \frac{d\varphi}{ds} + \frac{dy}{ds} \frac{d\theta}{ds}} \right) \bigg|_{z = z_0} - \text{sgn} \left( \sin \theta \frac{dx}{ds} \frac{d\varphi}{ds} + \frac{dy}{ds} \frac{d\theta}{ds} \right) \bigg|_{z = z_0} \pi.\]

The function \(\mu(t)\) in (4.6) must be so chosen that

\[\psi_1 - \psi_0 = \int_0^{t_1} \mu(t) dt, \quad \mu(t_1) = \mu(0) = 0.\]

The condition \(\mu(t_1) = \mu(0) = 0\) guarantees that the rotors will be at rest at times \(t = 0\) and 
\(t = t_1\).
3. Specify the law of motion \( s(t) \) in such a way that
\[
\dot{s}(t_1) = \dot{s}(t_2) = 0,
\]
and let the gyrostatic momentum evolve according to (4.7). In time \( t_2 - t_1 \) the sphere travels along the prescribed curve and stops at the end point.

4. Calculate the angle \( \psi_2 \) at the end point
\[
\psi_2 = \arctg \left( \frac{\sin \theta \frac{dy}{ds} \frac{d\varphi}{ds} - \frac{dx}{ds} \frac{dy}{ds} \frac{d\theta}{ds}}{\sin \theta \frac{dx}{ds} \frac{d\varphi}{ds} + \frac{dy}{ds} \frac{d\theta}{ds}} \right) \bigg|_{z = z_f} - \text{sgn} \left( \frac{\sin \theta}{\frac{dx}{ds} \frac{d\varphi}{ds} + \frac{dy}{ds} \frac{d\theta}{ds}} \right) \bigg|_{z = z_f} \pi.
\]

Rotate the sphere using (4.6), where \( \mu(t) \) satisfies the equations

\[
\psi_f - \psi_2 = \int_{t_2}^{T} \mu(t) \, dt, \quad \mu(t_2) = \mu(T) = 0.
\]

Thus, in time \( T \) the sphere is steered from an initial configuration to a desired final configuration.

Remark. To avoid these additional turns about the vertical, the spherical curve \( r(s) \) must be so chosen that its tangent is aligned with the tangent to the plane curve at the initial and final configurations.

5. EXAMPLES

As an example, let us calculate the gyrostatic momentum \( K(t) \) so that the point of contact follows 1) an arc of a circle and 2) an arc of a sinusoid. As the spherical curve we choose the great circle whose plane is orthogonal to the principal axis of inertia of the sphere \( e_3 \), that is,
\[
\gamma(s) = \left( \sin \frac{s}{R}, \cos \frac{s}{R}, 0 \right).
\]
Here the arc length parameter \( s \) varies from zero to its maximum value which is determined by the trajectory of the point of contact.

5.1. Rolling along a Circle

Suppose that the sphere rolls along a circle of radius \( \rho \); then
\[
r_{--}(s) = \left( -\rho \sin \frac{s}{\rho}, \rho \cos \frac{s}{\rho} \right), \quad s = 0 \ldots s_{\text{max}}.
\]
As already mentioned, on the sphere we choose the great circle orthogonal to \( e_3 \), thus
\[
\gamma(s) = \left( \sin \frac{s}{R}, \cos \frac{s}{R}, 0 \right).
\]

The function \( s(t) \) must obey (4.10), so we put
\[
s(t) = s_{\text{max}} \sin^2 \left( \frac{\pi}{2} t \right), \quad t \in [0, 1].
\]
Therefore, the control input \( K(t) \) takes the form
\[
K(t) = -\frac{s_{\text{max}} \pi}{2} \sin(\pi t) \left( \frac{I_1}{\rho} \sin \frac{s(t)}{R}, \frac{I_2}{\rho} \cos \frac{s(t)}{R}, \frac{I_3 + D}{R} \right),
\]
where \( s(t) \) is given by (5.4). The graph of \( K(t) \) for the case where a sphere of radius \( R = 1 \) rolls along a circle of radius \( \rho = \frac{1}{2} \) is shown in Fig. 3.
HOW TO CONTROL CHAPLYGIN’S SPHERE USING ROTORS

5.2. Rolling along a Sinusoidal Path (Slalom)

We want the point of contact to follow $N$ periods of the sinusoid

$$\mathbf{r}_{-}(-\tau) = (-\rho \sin \tau, \nu \tau), \quad \tau = 0 \ldots 2\pi N,$$

where $\rho$ is and $\nu$ are the amplitude and the frequency. The arc length now reads

$$s(\tau) = \sqrt{\nu^2 + \rho^2} E \left( \tau, \frac{\rho}{\sqrt{\nu^2 + \rho^2}} \right).$$

Here

$$E(\phi, k) = \int_{0}^{\phi} \sqrt{1 - k^2 \sin^2 \theta} \, d\theta$$

is the elliptic integral of the second kind.

Therefore, it seems reasonable (to avoid elliptic integrals) to parameterize all curves involved by $\tau$ instead of $s$. Writing $\mathbf{r}_{-}(-\tau)$ for the in-plane curve, $\gamma(\tau)$ — for the spherical curve and using this in (4.7), we get

$$\mathbf{K}(t) = -\dot{\tau} \left( (\mathbf{I} + D\mathbf{E}) \left( \frac{d\gamma}{d\tau} \times \gamma \right) + (\tilde{k}_p - \tilde{k}_b) I\gamma \right),$$

$$\tilde{k}_p = \left( \frac{d^2 x}{d\tau^2} - \frac{d x \, d^2 y}{d\tau^2} \right) / \left( \frac{ds}{d\tau} \right)^2, \quad \tilde{k}_b = R^2 \left( \frac{d\gamma}{d\tau} \times \frac{d^2 \gamma}{d\tau^2} \right) / \left( \frac{ds}{d\tau} \right)^2.$$

Plugging (5.3) and (5.6) into (5.9), we obtain

$$\mathbf{K}(t) = \dot{\tau} \left( \frac{\nu \rho \sin \tau}{\nu^2 + \rho^2 \cos^2 \tau} I_1 \sin \frac{s(\tau)}{R}, \frac{\nu \rho \sin \tau}{\nu^2 + \rho^2 \cos^2 \tau} I_2 \cos \frac{s(\tau)}{R}, -\sqrt{\nu^2 + \rho^2 \cos^2 \tau} \frac{I_3 + D}{R} \right),$$

where $s(\tau)$ is is given by (5.7). Similar to (5.4) define $\tau(t)$ as

$$\tau(t) = 2\pi N \sin^2 \left( \frac{\pi t}{2} \right), \quad t \in [0, 1].$$

For $N = 2$ the graph of $\mathbf{K}(t)$ is shown in Fig. 4.
Fig. 4. The graph of the gyrostatic momentum $K$ versus time; this input steers the sphere of radius $R = 1$ along the sinusoid $x = 2 \sin y$, $y \in [0 \ldots 4\pi]$, here $I = \text{diag}(2, 3, 4)$, $D = 10$.

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