Rolling of a Rigid Body Without Slipping and Spinning: Kinematics and Dynamics

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Abstract
In this paper we investigate various kinematic properties of rolling of one rigid body on another both for the classical model of rolling without slipping (the velocities of bodies at the point of contact coincide) and for the model of rubber-rolling (with the additional condition that the spinning of the bodies relative to each other be excluded). Furthermore, in the case where both bodies are bounded by spherical surfaces and one of them is fixed, the equations of motion for a moving ball are represented in the form of the Chaplygin system. When the center of mass of the moving ball coincides with its geometric center, the equations of motion are represented in conformally Hamiltonian form, and in the case where the radii of the moving and fixed spheres coincide, they are written in Hamiltonian form.

Keywords
Rolling without slipping
Nonholonomic constraint
Chaplygin system
Conformally Hamiltonian system

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1 Introduction

We consider some kinematical and dynamical problems for a rigid body moving on a fixed surface. We assume that interaction between the body and the surface is realized by means of the nonholonomic constraint which corresponds to the conditions that velocity of the contact point and projection of the body angular velocity to the surface normal vanish. In systems with rolling as a rule, only the first condition is taken. It fixes the model of an absolutely rough surface. Several interested problems of this kind were solved by Appell, Routh, Chaplygin and Woronetz (a survey of these results can be found in [1–3]). The second condition can be obtained under the assumption of a large resistance to the twist during the motion. It is also mentioned in classical works. In [4,5]

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J. Hadamard and A. Beghin studied kinematics and dynamics of systems with zero twist. They also indicated conditions under which the constraint turns to be holonomic. Results from [4, 5] were rediscovered by modern researchers [6–8]. Note also that the model we deal with below is, in some aspects, simpler than the classical one. This fact stimulated its research in problems of robotics, control, etc. (for the case of a moving ball discussion of the corresponding literature is contained in [9]). In [10–12] several problems of rolling of a rigid body on a surface are considered and the term “rubber rolling” is proposed, stressing physical nature of the no-twist condition.

In this paper we discuss systematically kinematical problems for the motion of a rigid body in the case of rubber rolling. Complete proofs are presented. We also present dynamical equations in quasi-coordinates (in a form, convenient for further study) in the case when both the moving and fixed bodies are bounded by spherical surfaces. The equations of motion are presented in the form of Chaplygin equations. We point out the cases when the dynamics of the system is Hamiltonian (may be, after an appropriate time change).

2 Two constraints

2.1 Rolling without slipping

Let $B_1$ and $B_2$ be two absolutely rigid bodies in the Euclidean space $\mathbb{R}^3$. We assume that the boundaries $S_1 = \partial B_1$ and $S_2 = \partial B_2$ are smooth (co)oriented surfaces, and for some constant $c$ the normal curvatures of the surface $S_1$ are strictly greater than $c$, and the normal curvatures of the surface $S_2$ are strictly greater than $-c$. This condition is satisfied, for example, if both bodies are strictly convex (then one can take $c = 0$) or one of the bodies is a plane, while the other is strictly convex.

In this situation the problem of rolling without slipping of the body $B_2$ on $B_1$ (which we shall regard as fixed for definiteness) is well-defined. In this case the rolling without slipping means that the velocity of the contact point $Q_2 = Q_2(t) \in B_2$ vanishes at any instant of time $t$.

The condition of rolling without slipping is treated in mechanics as a constraint linear in velocities. In this section we introduce some geometric considerations regarding integrability of this and another similar constraint, for which the no-slip condition is supplemented with the no-twist condition.

Assume that the body $B_2$ is convex and compact. Then $S_2$ is diffeomorphic to the two-dimensional sphere, and $\mathcal{P} = S_1 \times SO(3)$ is the configuration space of the system. Indeed, any position of the system is uniquely defined by the orientation of $B_2$ (an element of $SO(3)$) and the contact point $Q_1 \in S_1$.

Recall that a constraint is called completely integrable if it is equivalent to certain restrictions on the configuration space, i.e. to equations of the form $f = \text{const}$, where $f: \mathcal{P} \to \mathbb{R}^k$ is a smooth function, $k \geq 1$. A constraint is called completely nonintegrable if it does not generate any restriction on the configuration space. In all intermediate situations we speak of a partially integrable constraint.

Let $\Pi$ denote the common tangent plane to $S_1$ and $S_2$ at the contact point. We start with a well-known statement.

**Proposition 1.** The condition of rolling without slipping presents a completely nonintegrable constraint.
Proof. It is sufficient to prove that, without violating the constraint, the body $B_2$ can be rolled from one arbitrary position $(p_- \in \mathcal{P})$ into another, also arbitrary position $p_+ \in \mathcal{P}$. Let

$$S_1 \ni Q_1^\pm = Q_2^\pm \in S_2$$

be the points of contact of the bodies in the positions $p_\pm$. We draw smooth curves $\gamma_1$ and $\gamma_2$ on the surfaces $S_1$ and $S_2$ in such a manner that

- $\gamma_1$ starts at the point $Q_1^-$ and ends at $Q_1^+$,
- $\gamma_2$ starts at the point $Q_2^-$ and ends at $Q_2^+$,
- the lengths of the curves $\gamma_1$ and $\gamma_2$ coincide.

We roll the body $B_2$ on $B_1$ without slipping in such a way that the curves $\gamma_1$ and $\gamma_2$ have all the time a common tangent at the contact point $S_1 \ni Q_1(t) = Q_2(t) \in S_2$ between the bodies. To do so, it is necessary to move the body $B_2$ at any instant of time $t$ with angular velocity $\omega(t)$ perpendicular to the common tangent to the curves $\gamma_1$ and $\gamma_2$. The projection $\omega(t)$ onto the normal to the plane $\Pi \equiv \Pi(t)$ can be calculated with the help of Proposition 4 (see below).

The no-slip condition implies that the motion starting at the contact point $Q_1^-$ on $B_1$ and at $Q_2^-$ on $B_2$ ends at some instant of time $t_+$ in a position where the bodies touch each other at the points $Q_1^+$ and $Q_2^+$. To bring the system into the required position $p_+$, we only need to turn $B_2$ through the required angle around the normal to $\Pi(t_+)$. 

2.2 Rolling without slipping and twisting

Now consider another constraint where, in addition to the no-slip condition, the no-twist condition is imposed at the point of contact, that is at any instant of time $t$ the angular velocity $\omega(t)$ is parallel to the plane $\Pi(t)$. We call this constraint rolling without twisting.

Assume that the bodies $B_1$ and $B_2$ are equal as geometric figures. This means that there exists a distance-preserving bijection $b: B_1 \rightarrow B_2$. There can be several such bijections. We fix one of them.

The following simple observation is also well known, see, e.g., [7,13].

Proposition 2. Assume that at the initial instant of time $b$ coincides with the symmetry about the plane $\Pi$. Then this property holds at any instant of time in the process of motion (rolling without twisting).

The proof will follow from Proposition 4.

If $B_1$ and $B_2$ are equal convex figures of a general form, by Proposition 2 there is a surface $\mathcal{S}$ of codimension 3 in the phase space $\mathcal{P}$ such that any motion compatible with the constraint and starting on $\mathcal{S}$, stays on $\mathcal{S}$.

If $B_1$ and $B_2$ are balls of the same radius, then the constraint is completely integrable. Indeed, $\mathcal{P} = S^2 \times SO(3)$ can be regarded here as the phase space, where the first factor can be treated as the normal $n$ to the sphere $S_1$ at the point of contact and the second one as the orthogonal matrix $A$ prescribing the orientation of the moving body $B_2$. Proposition 2 implies that $\mathcal{P}$ foliates into invariant integral surfaces of the form

$$I_B = \{ (n, S \circ B) \in \mathcal{P} : n \in S^2 \}, \quad B \in SO(3),$$
where \( S_n \in SO(3) \) is the symmetry about the plane perpendicular to \( n \). Each of the surfaces \( I_B \) is diffeomorphic to the two-dimensional sphere and is a graph over the first factor in the direct product \( S^2 \times SO(3) \).

In addition to the case of two identical balls and the partial integrability indicated in Proposition 2, there are also other cases of absence of complete nonintegrability. For example, the rolling without twisting is a completely integrable constraint if \( B_1 \) and \( B_2 \) are cylinders. Conventional tools for study such effects are the Frobenius criterion and the Rashevsky–Chow theorem. The nonintegrability of the above constraint (also treated as controllability of the system) was established in [6] (rolling of a sphere on a plane or over a sphere of a different radius), [8] (rolling of an axisymmetric body on a plane), [7] (the case of strictly convex bodies). The most complete statements are contained in [13]. The methods used in these papers boil down to analysis of the vector fields which the constraint admits in the configuration space \( \mathcal{P} \) and of their commutators.

In what follows, with small additional restrictions, we comment on the nonintegrability of the constraint by explicitly constructing a control relocating the system from one arbitrary position into another. These arguments are conceptually close to traditional ones but probably, are more geometric and constructive.

**Proposition 3.** Assume that for any pair of points \( Q_1 \in S_1 \) and \( Q_2 \in S_2 \) the Gaussian curvature \( K_1(Q_1) \) of the surface \( S_1 \) is strictly less than the Gaussian curvature \( K_2(Q_2) \) of the surface \( S_2 \). Then, by rolling without twisting one can relocate \( B_2 \) from any position \( p_- \in \mathcal{P} \) into any other position \( p_+ \in \mathcal{P} \).

### 3 Kinematic properties of rolling

For the proof of Propositions 2 and 3 we need some facts from kinematics. To analyze the motion, it is convenient to pass to a moving reference frame attached to the tangent plane \( \Pi(t) \). In this system the bodies \( B_1 \) and \( B_2 \) roll on the (now fixed) plane \( \Pi \), touching \( \Pi \) at coincident points. If \( B_1 \) and \( B_2 \) are convex, then in the process of motion they are on different sides of \( \Pi \). We fix the orientation of \( \Pi \), e.g., by choosing the unit normal \( e_z \perp \Pi \) directed to the inside of the body \( B_1 \) and to the outside of \( B_2 \). This choice of a “positive” normal prescribes the orientation on the surfaces \( S_1 \) and \( S_2 \).

The contact point \( Q_i(t) \in S_i \) of the surfaces \( S_i \) and \( \Pi \) leaves on \( S_i \) a trace, the oriented curve \( \gamma_i \). A similar curve \( \gamma \) also appears on the plane \( \Pi \). Let \( s \) be a natural parameter (arc length) on \( \gamma_i \). The no-slip condition implies that \( s \) is also a natural parameter on \( \gamma_2 \) and \( \gamma \). Let \( k_i = k_i(s) \), \( i = 1, 2 \) be the geodesic curvature of \( \gamma_i \) as a curve on the oriented surface \( S_i \) and let \( k = k(s) \) be the curvature of \( \gamma \).

**Proposition 4.** Let \( \dot{s} \) be the velocity of the point of contact for rolling without slipping. Then

\[
k(s) = k_i(s) + \omega_i / \dot{s}, \quad \omega_i = (\omega, e_i), \quad i = 1, 2.
\]

**Corollary 5.** Let the initial points \( \gamma_1(0), \gamma_2(0) \) and \( \gamma(0) \) be fixed. Then for rolling without slipping and twisting, i.e. for \( \omega_z = 0 \), any of the curves \( \gamma_1, \gamma_2, \gamma \) uniquely defines the other two and uniquely defines the motion of a system in a natural parametrization.

**Corollary 6.** Assume that in the case of rolling without twisting one of the curves \( \gamma_1, \gamma_2, \gamma \) is a geodesic. Then the other two are geodesics also.
As another corollary we obtain the proof of Proposition 2, since by Proposition 2 we have $\gamma_2 = b(\gamma_1)$.

Assume that in the process of motion the point of contact $Q_2(t), t \in [t_-, t_+]$ has moved on $S_2$ along the closed curve $\gamma_2 = \gamma_2(s), s \in [s_-, s_+]$. Then in the reference frame attached to $\Pi$, the relocation of $B_2$ amounts to a translation by the vector $Q(s_-)Q(s_+)$ and to a rotation through the angle $\alpha$ about the vector $e_\varepsilon$.

**Proposition 7.** For rolling without slipping

$$\alpha = -\Omega + \int_{t_-}^{t_+} \omega(t) dt, \quad \Omega = \int_{s_-}^{s_+} k_2(s) ds. \tag{1}$$

The quantity $\Omega$ is a solid angle for the Gaussian mapping $\mathcal{D}_2 \to S^2$, where $\mathcal{D}_2 \subset S_2$ is a domain bounded by the curve $\gamma_2$ and $S^2$ is a unit sphere. By the Gauss–Bonnet theorem $\Omega = \int_{\mathcal{D}_2} K_2 d\sigma$, where $K_2$ is the Gaussian curvature of the surface $S_2$.

The proof of Propositions 4 and 7 can be found, for example, in [14].

Let $c_2 \subset S_2$ be a circle of small radius $r$. This implies that the points of the closed curve $c_2$ are equidistant from some point $O_2 \in S_2$ in a metric induced on $S_2$ from $\mathbb{R}^3$.

The idea of proving Proposition 3 is as follows. Since the trace $\gamma_2$ uniquely defines the trajectory of motion of the system in the space $\mathcal{D}$, it is sufficient to point out the oriented curve $\gamma_2$ defining the necessary motion. We compose the curve $\gamma_2$ from several smooth curves. All of them (except, perhaps, for the first one) are circles. The main stages of motion, namely:

1. transfer of the point $Q_2^-$ to $Q_2^+$; as a result, $S_2$ begins to touch the surface $S$ at the required point $Q_2 = Q_2^+ \in S_2$;

2. transfer of the point $Q_1^-$ to $Q_1^+$ (in such a way that the point $Q_2$ gets into $Q_2^+$) again; as a result, both points of contact $Q_1 \in S_1$ and $Q_2 \in S_2$ find themselves at the required locations;

3. motion the result of which is equivalent to a turn around the normal to $\Pi$ at the point $Q_1^+ = Q_2^+ \in \Pi$,

can be easily calculated explicitly using formulas of spherical trigonometry provided that $S_1$ and $S_2$ are spheres.

In what follows we deal with oriented circles $c_2 \subset S_2$ with a marked point. Any such circle is uniquely defined by the marked point $Q_2$, the radius $r$, the unit vector $e_2 \in T_{Q_2}S_2$ pointing the direction of the shortest geodesic from $Q_2$ to $O_2$ and the orientation sign $\sigma \in \{+, -\}$, which we shall regard as positive if $c_2$ prescribes counterclockwise rotation as viewed from the end of the vector of the outer normal to $S_2$:

$$c_2 = e_2^\sigma(Q_2, r, e_2).$$

Let us fix the position of the system $p \in \mathcal{D}$. Then the contact points $Q_1 = Q_1(p) \in S_1$ and the linear isometric isomorphism $R_p: T_{Q_1}S_2 \to T_{Q_1}S_1$ are fixed. Consider also two turns $J^\pm: T_{Q_1}S_1 \mapsto T_{Q_1}S_1$ of the two-dimensional plane $T_{Q_1}S_1$ around the point $Q_1$ through the angle $\pm \pi/2$:

$$J^\pm v = \pm e_\varepsilon \times v, \quad v \in T_{Q_1}S_1.$$
Consider the motion of the system where $S_2$ rolls without twisting on $S_1$ in such a way that the trace $\gamma$ turns out to be the circle $c_2^*(Q_2(p), r_2, e_2)$. Generally speaking, the corresponding curve $\gamma_1$ will not be closed, so that there arises the map

$$ (p, \sigma, r_2, e_2) \mapsto \hat{Q}_1 = F^\sigma_p(r_2, e_2) \in S_1, $$

where $\hat{Q}_1$ is the finite point of the curve $\gamma_1$.

**Lemma 8.** Let $K_i$ be the Gaussian curvature of the surface $S_i$ at the point $Q_i$, $i = 1, 2$. Then

$$ \left. \frac{d}{dr} \right|_{r=0} F^\sigma_p = \left. \frac{d^2}{dr^2} \right|_{r=0} F^\sigma_p = 0, \quad \left. \frac{d^3}{dr^3} \right|_{r=0} F^\sigma_p(r, e_2) = 6\pi(K_2 - K_1)J^\sigma R_p e_2. \quad (2) $$

**Corollary 9.** In the case $K_1 \neq K_2$ the map $(r, e) \mapsto F^\sigma_p(r^{1/3}, e)$ is a local $C^1$-diffeomorphism of the neighborhood of zero (point $r = 0$) of the plane $\Pi$ with polar coordinates $(r, e)$ onto a neighborhood of the point $Q_1$ in $S_1$.

**Lemma 10.** The geodesic curvature $k$ of the curve $c_2 = c_2^*(Q_2, r, e)$ at any point is equal to

$$ k = r^{-1}(1 - K_2 r^2/3 + O(r^3)), \quad (3) $$

where $K_2 = K_2(Q_2)$ is the Gaussian curvature at the point $Q_2$.

The length of the circle $c_2$ is

$$ |c_2| = 2\pi r(1 - K_2 r^2/6 + O(r^3)). \quad (4) $$

The proof of Lemmas 8 and 10 is contained in Section 6.

### 4 Proof of Proposition 3

1. The contact points $Q_2^- = Q_2^-$ and $Q_2^+ = Q_2^+$ in the positions $p_-$ and $p_+$ may be thought of as coincident. Indeed, to do so, it is sufficient to roll $B_2$ in an arbitrary way, without violating the constraint, into a position with the contact point $Q_2^+$ and to regard this position $p_-$ as initial instead of $p_-$.

2. The contact points $Q_1^- = Q_1^-$ and $Q_1^+ = Q_1^+$ may also be thought of as coincident. Indeed, to roll the ball $B_2$ from the position $\hat{p}_-$ into the position $\hat{p}_+$, when $Q_1^- \in B_1$ and $Q_1^+ \in B_2$ are the contact points, we proceed as follows. Consider the rolling without twist uniquely defined (Corollary 5) by the trace

$$ \gamma = c_2^*(Q_2, r, e), \quad Q_2 = Q_2^- = Q_2^+. $$

By Corollary 9 any point $\hat{Q}_1$ sufficiently close to $Q_1^-$ can turn to be the finite point of the trace $\gamma$. Hence, within a finite number of such steps the contact point $Q_1$ can be relocated into any position on $S_1$.

3. Thus, $Q_1^- = Q_1^+$ and $Q_2^- = Q_2^+$, so that we only need to turn the body $B_2$ around the normal $e_z$ to the tangent plane $\Pi$ through some angle $\alpha$. This can be done as follows. First we roll $S_2$ on

\[\text{It is useful to keep in mind that } K_2(Q_2) = K_2(Q_2) + O(r).\]
we have: deal with one of the simplest cases where the supporting surface

5.1 Equations of motion and first integrals

5 Rolling without twisting in the case of spherical contact

By Lemma 10

\[
\alpha_2 = r^{-1}(1 - \frac{K_2 r^2}{3} + o(r^3))|c_2| + \frac{1}{r^2}(1 - \frac{K_2 r^2}{3} + o(r^3))|\hat{c}_2|
\]

\[
= 2\pi(1 - \frac{K_2 r^2}{3})(1 - \frac{K_2 r^2}{6}) + o(r^3) + 2\pi(1 - \frac{K_2 r^2}{3})(1 - \frac{K_2 r^2}{6}) + o(r^3)
\]

\[
= 4\pi - 2\pi K_2 r^2 + o(r^3).
\]

Analogously, \( \alpha_1 = 4\pi - 2\pi K_1 r^2 + o(r^3) \). Since \( K_1 \neq K_2, B_2 \) can be turned in this way relative to \( B_1 \) around \( e_5 \) through any small angle \( \alpha \). This proves Proposition 3.

5 Rolling without twisting in the case of spherical contact

5.1 Equations of motion and first integrals

Now consider some dynamical problems in the case of rolling without slipping and twisting. We deal with one of the simplest cases where the supporting surface \( S \) is a sphere over which the body \( B \), also bounded by the spherical surface \([10, 15]\), rolls. Fig.1 shows three possible scenarios of rolling. We refer to the center of the spherical shell of the body in the cases depicted in Fig.1(a), (b) and the geometric center of the cavity in the case depicted in Fig.1(c) as the geometric center of the body \( O_0 \). The configuration space of the system is \( \mathcal{M} = S^2 \times SO(3) \), where the first factor corresponds to a possible position of the center \( O_0 \) and the second one corresponds to the orientation of the body.

Choose a coordinate system \( Cxyz \) rigidly attached to a moving body with origin at the body’s center of mass. In what follows all vectors are assumed to be given in these axes. We define the unit normal vector at the contact point \( n \), directed towards the rolling body, and the orthogonal matrix

\[
Q = \begin{pmatrix}
\alpha_1 & \beta_1 & \gamma_1 \\
\alpha_2 & \beta_2 & \gamma_2 \\
\alpha_3 & \beta_3 & \gamma_3
\end{pmatrix},
\]

which prescribes the orientation of the body. The columns of the matrix present the coordinates of the fixed unit vectors \( \alpha, \beta \) and \( \gamma \) in the moving axes. The pair \( (n, Q) \in \mathcal{M} \) completely determines the position of the system.
Three possible scenarios of rolling.

Denote the radii of curvature taking into account the sign for the fixed surface $S$ and the boundary of the body $\partial B$ by $a$ and $b$, respectively (see Fig. 1). Let $c$ denote the vector from the geometric center $OB$ to the body’s center of mass $C$ (it is constant in the chosen coordinate system) and let $r$ denote the vector from the center of mass to the point of contact. Let $\omega$ and $v$ be the angular velocity and the velocity of the body’s center of mass $C$. Then the no-slip and no-twist conditions can be represented as

$$v + \omega \times r = 0, \quad (\omega \cdot n) = 0, \quad r = -bn - c.$$  (5)

Note that $b > 0$ for rolling in Fig. 1(a), (b) and $b < 0$ for that in Fig. 1(c).

Equations of motion for this system can be obtained by the method of undetermined multipliers (see, e.g., [10]) and have the form

$$\bar{I}\dot{\omega} = (I\omega) \times \omega - mr \times (\omega \times r) + \lambda n + M_Q, \quad \dot{n} = kn \times \omega, \quad \dot{Q} = \bar{\omega}Q,$$

$$\bar{I} = I + mr^2E - mr \otimes r, \quad k = \frac{a}{a+b}, \quad \lambda = -\frac{(I\omega \times \omega - mr \times (\omega \times r) + M_Q, I^{-1}n)}{(n, I^{-1}n)},$$  (6)

where $I$ and $m$ are the inertia tensor and the mass of the body, $E$ is the unit matrix, $M_Q$ is the moment of external forces relative to the point of contact, $\bar{\omega}$ is the skew-symmetric matrix of the angular velocity, whose components are given by $\bar{\omega}_{ij} = \epsilon_{ijk}\omega_k$, the value and the sign $k$ depend on the rolling scenario and are indicated in Fig. 1.

As was shown above, in the case where a ball rolls over a ball of the same radius, i.e. at $k = 1/2$, the system of constraints (5) becomes holonomic. As a result, the system (6) admits geometric integrals which can be written in matrix form as

$$B = S_nQ = \text{const.}, \quad S_n = E - 2n \otimes n.$$
where \( \mathbf{B} \) is the orthogonal matrix, \( \mathbf{S}_n \) is the symmetry about the plane orthogonal to \( \mathbf{n} \). These integrals allow one to represent the system in Lagrangian and Hamiltonian form on \( S^2 \) without a change of time (see below).

We assume that the external forces are potential with the potential \( U(\mathbf{n}) \) depending only on the vector \( \mathbf{n} \). Then the equations for the vectors \( \mathbf{\omega} \) and \( \mathbf{n} \) separate. We rewrite them in a form which is more convenient for further analysis. To this end, using the equalities \( (\mathbf{\omega}, \mathbf{n}) = (\dot{\mathbf{\omega}}, \mathbf{n}) = 0 \), we obtain the equations

\[
\mathbf{I}\dot{\mathbf{\omega}} = (\mathbf{J} + \Lambda \mathbf{E})\mathbf{\omega} - mb(\mathbf{\omega}, \mathbf{c})\mathbf{n}, \quad \mathbf{I}\dot{\mathbf{n}} = (\mathbf{J} + \Lambda \mathbf{E})\mathbf{n} - mb(\mathbf{\omega}, \mathbf{c})\mathbf{\omega},
\]

\[
\mathbf{J} = \mathbf{I} + m(b^2 + c^2)\mathbf{E} - mc \otimes \mathbf{c}, \quad \Lambda = 2mb(\mathbf{c}, \mathbf{n}).
\]

Using them, we represent the equations of motion for the vectors \( \mathbf{\omega} \) and \( \mathbf{n} \) as:

\[
\tilde{\mathbf{J}}\dot{\mathbf{\omega}} = (\tilde{\mathbf{J}}\mathbf{\omega}) \times \mathbf{\omega} - \frac{mb}{k}(\mathbf{\omega}, \mathbf{c})\mathbf{n} - mbc \times (\mathbf{\omega} \times \mathbf{n}) + \tilde{\lambda}\mathbf{n} + kn \times \frac{\partial U}{\partial \mathbf{n}}, \quad \mathbf{n} = kn \times \mathbf{\omega},
\]

\[
\tilde{\lambda} = - \frac{(\tilde{\mathbf{J}}\mathbf{\omega}) \times \mathbf{\omega} - \frac{mb}{k}(\mathbf{\omega}, \mathbf{c})\mathbf{n} - mbc \times (\mathbf{\omega} \times \mathbf{n}) + kn \times \frac{\partial U}{\partial \mathbf{n}}}{(\mathbf{n}, \tilde{\mathbf{J}}^{-1}\mathbf{n})},
\]

where \( \tilde{\mathbf{J}} = \mathbf{J} + \Lambda \mathbf{E} \). Below we shall assume that the axes of the coordinate system \( \mathbf{Cxyz} \) are directed along the eigenvectors of the matrix \( \mathbf{J} \), i.e. \( \mathbf{J} = \text{diag}(J_1, J_2, J_3) \).

These equations are analogous to the Euler–Poisson equations in the dynamics of a rigid body with a fixed point. In what follows (unless otherwise stated) we discuss the properties only of this six-dimensional reduced system.

Remark 1. If we let the radius of the fixed sphere \( S \) in the cases in Fig.1(a), (b) tend to infinity \( (a \rightarrow \infty) \), then in the limit we obtain the rolling of the ball on the plane \( \mathbf{k} = 1 \).

The system (7) admits two general first integrals

\[
F_0 = \mathbf{n}^2, \quad F_1 = (\mathbf{\omega}, \mathbf{n}),
\]

whose physical values are equal to \( F_0 = 1, F_1 = 0 \). On the level subspace \( F_1 = 0 \) there also exists the energy integral

\[
E = \frac{1}{2}(\mathbf{\omega}, \tilde{\mathbf{J}}\mathbf{\omega}) + U(\mathbf{n}).
\]

We restrict Eqs. (7) to the manifold

\[
\mathcal{M}^4 = \{(\mathbf{\omega}, \mathbf{n}) \mid \mathbf{n}^2 = 1, (\mathbf{\omega}, \mathbf{n}) = 0\},
\]

which is diffeomorphic to \( TS^2 \). We obtain a four-dimensional system possessing the energy integral (9). The properties of this system depend essentially on whether it possesses any additional (tensor) invariants \([1–3]\). In this case these can be an additional first integral and an invariant measure. The system possessing them is conformally Hamiltonian and integrable by the Euler–Jacobi theorem, hence, its behavior is regular, see \([16]\) for more details. But if the existence of a measure and the Euler–Jacobi integrability is excluded, then in the presence of an integral the behavior is also regular \([17]\). Otherwise the system can display chaotic behavior typical for both conservative and dissipative systems.
5.2 Equations of motion in the Chaplygin form

Let us write Eqs. (7) in local variables on the sphere \( n^2 = 1 \). To this end we choose the spherical coordinates
\[
n_1 = \sin \theta \sin \varphi, \quad n_2 = \sin \theta \cos \varphi, \quad n_3 = \cos \theta,
\]
and express the angular velocity on the level subspace \((\omega, n) = 0:\)
\[
k \omega = \dot{n} \times n = (\dot{\theta} \cos \varphi - \dot{\varphi} \cos \theta \sin \theta \sin \varphi, \quad -\dot{\theta} \sin \varphi - \dot{\varphi} \cos \theta \sin \theta \cos \varphi, \quad \dot{\varphi} \sin^2 \theta).\]

Define the appropriate Lagrangian
\[
L = T(\theta, \varphi, \dot{\theta}, \dot{\varphi}) - U(\theta, \varphi),
\]
\[
T = \frac{1}{2}(\omega, \dot{\omega}) = \frac{1}{2k^2}(J_1(\dot{\theta} \cos \varphi - \dot{\varphi} \cos \theta \sin \theta \sin \varphi)^2 + J_2(\dot{\theta} \sin \varphi + \dot{\varphi} \cos \theta \sin \theta \cos \varphi)^2 + J_3 \sin^4 \dot{\varphi} + \Lambda(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2 \theta)),
\]
\[
\Lambda = 2mb(n, c) = 2mb(c_1 \sin \theta \sin \varphi + c_2 \sin \theta \cos \varphi + c_3 \cos \theta).
\]

Differentiating by virtue of the system (7), one can show that the equations of motion are represented in the form of the Chaplygin system
\[
\left(\frac{\partial L}{\partial \theta}\right) - \frac{\partial L}{\partial \dot{\theta}} = \theta S, \quad \left(\frac{\partial L}{\partial \varphi}\right) - \frac{\partial L}{\partial \dot{\varphi}} = -\dot{\varphi} S,
\]
\[
S = \frac{1 - 2k}{2k^3} \sin \theta \left(\frac{\partial G}{\partial \theta} \sin \theta \dot{\varphi} - \frac{1}{\sin \theta} \frac{\partial G}{\partial \varphi}\right),
\]
\[
G = (n, \dot{n}, n) - \Lambda
\]
\[
= (J_1 \sin^2 \varphi + J_2 \cos^2 \varphi) \sin^2 \theta + J_3 \cos^2 \theta - 2mb((c_1 \sin \varphi + c_2 \cos \varphi) \sin \theta + c_3 \cos \theta).
\]

Thus, as indicated above, for \( k = 1/2 \) and an arbitrary \( c \) the system is represented in Lagrangian (and, consequently, in Hamiltonian) form.

The property of being Hamiltonian at \( k = 1/2 \). Using the formulas for the angular momentum of a material point on the surface of a sphere
\[
M_1 = -\rho_\varphi \frac{\sin \varphi \cos \theta}{\sin \theta} + \rho_\theta \cos \varphi, \quad M_2 = -\rho_\varphi \frac{\cos \varphi \cos \theta}{\sin \theta} - \rho_\theta \sin \varphi, \quad M_3 = -\rho_\varphi,
\]
where \( \rho_\varphi = \partial L/\partial \dot{\varphi}, \rho_\theta = \partial L/\partial \dot{\theta} \), we obtain
\[
M = 2n \times (J \omega \times n).
\]

This allows us to represent the equations of motion in Hamiltonian form on the (co)algebra (8)
\[
\dot{M} = M \times \frac{\partial H}{\partial M} + n \times \frac{\partial H}{\partial n}, \quad \dot{n} = n \times \frac{\partial H}{\partial n},
\]
where the Poisson brackets are given by
\[
\{M_i, M_j\} = \epsilon_{ijk} M_k, \quad \{M_i, n_j\} = \epsilon_{ijk} n_k, \quad \{n_i, n_j\} = 0,
\]
and the Hamiltonian is
\[
H = \frac{1}{8\rho} \left( (\text{Tr} J - (n, Jn) + \Lambda) M^2 - (M, JM) \right), \quad \rho = \det(J(n, J^{-1}n)).
\]
The property of being conformally Hamiltonian at $c = 0$. In this case $\Lambda = 0$, and due to the existence of an invariant measure by the Chaplygin theorem [18] (see in details [1]) the system is represented in conformally Hamiltonian form (i.e. it is Hamiltonian after the change of time). By using Eq. (11), we obtain:

$$M = \rho_0^{-1+\frac{1}{k}} n \times (J\omega \times n), \quad \rho_0 = \det (n, J^{-1} n).$$

Here

$$\dot{M} = \rho_0^{-1+\frac{1}{k}} (M \times \frac{\partial H}{\partial M} + n \times \frac{\partial H}{\partial n}), \quad \dot{n} = \rho_0^{-1+\frac{1}{k}} n \times \frac{\partial H}{\partial M},$$

$$H = \frac{k^2}{2\rho_0^{-1+\frac{1}{k}}} ((\text{Tr} J - (n, J n))M^2 - (M, J M)).$$

Remark 2. In [1] the system (5.3) with $c = 0$ was presented in conformally Hamiltonian form, which differs from this one.

6 Appendix: proof of auxiliary statements

Proof. [of Lemma 8] By Proposition 4 the geodesic curvatures of the curves $\gamma_1$ and $\gamma_2$ coincide at the corresponding points and in view of Lemma 10 are equal to

$$r_2^{-1}(1 - K_2 r_2^2/3 + O(r_2^3)).$$

This means that $\gamma_1$ is up to $O(r_2^2)$ an arc of the circle $c_1 = c_1(Q_1, r_1, e_1) \subset S_1$, where $e_1 = R_\rho e_2$ and $r_1$ satisfies the equation

$$r_1^{-1}(1 - K_1 r_1^2/3 + O(r_1^3)) = r_2^{-1}(1 - K_2 r_2^2/3 + O(r_2^3)).$$

(12)

Here $K_1$ is the Gaussian curvature at the point $Q_1$. Equation (12) implies

$$r_1 = r_2 + r_2^3(K_2 - K_1)/3 + O(r_2^4).$$

The length of the arc coincides with the length of the circle $c_2$, where by (4) $|c_2| = 2\pi r_2(1 - K_2 r_2^2/6 + O(r_2^4))$. Analogously length of the circle $c_1$ equals

$$|c_1| = 2\pi r_1(1 - K_1 r_1^2/6 + O(r_1^3)) = |c_2| + \pi r_2^3(K_2 - K_1) + O(r_2^4).$$

This implies Eq. (2).

Proof. [of Lemma 10]. There exist local coordinates $(x, y)$ on $S_2$ such that the center $O_2$ of the circle $c_2$ has coordinates $(0, 0)$ and the metric on $S_2$ is

$$ds^2 = (1 - K_2(x^2 + y^2)/2 + O_3(x, y))(dx^2 + dy^2),$$

where $O_3(x, y)$ are terms of order, at least 3 in $x, y$. It is easy to obtain such coordinates $x, y$ from isothermal coordinates by a small additional normalization.
It is convenient to pass to polar coordinates $\rho^2 = x^2 + y^2$, $\tan \phi = y/x$. Then
\[ds^2 = g_{11}d\rho^2 + 2g_{12}d\rho d\phi + g_{22}d\phi^2 = (1 - K_2 \rho^2/2 + \mathcal{O}(\rho^4))(d\rho^2 + \rho^2 d\phi^2).\]

Christoffel symbols have the form
\[
\Gamma^1_{11} = -K_2 \rho + \mathcal{O}(\rho^2), \quad \Gamma^1_{12} = \mathcal{O}(\rho^3), \quad \Gamma^1_{22} = -\rho (1 - K_2 \rho^2/2 + \mathcal{O}(\rho^3)), \\
\Gamma^1_{11} = \mathcal{O}(\rho), \quad \Gamma^2_{12} = \rho^{-1}(1 - K_2 \rho^2/2 + \mathcal{O}(\rho^3)), \quad \Gamma^2_{22} = \mathcal{O}(\rho^3).
\]

Straightforward calculation shows that the geodesics passing through the point $(0,0)$, are determined by the equation $\phi = \phi_0 + \mathcal{O}(\rho^2)$. Therefore
\[c_2 = \{(\rho, \phi) : \rho - K_2 \rho^3/12 + \mathcal{O}(\rho^4) = r\}.
\]
Equation (4) follows from the following one
\[\rho = r + K_2 r^3/12 + \mathcal{O}(r^4).
\]

To compute the geodesic curvature of the curve $\rho(s), \phi(s)$, we use the Beltrami formula
\[
k = \Lambda \left[ \Gamma^2_{11} \rho'' - (2\Gamma^2_{12} - \Gamma^1_{11}) \rho' \phi'' + (\Gamma^2_{22} - 2\Gamma^1_{12}) \rho' \phi'^2 - \Gamma^2_{22} \phi'^3 + \rho' \phi'' - \rho'' \phi' \right],
\]
\[
\Lambda = (g_{11}g_{22} - g_{12}^2)^{1/2} (g_{11} \rho'^2 + 2g_{12} \rho' \phi' + g_{22} \phi'^2)^{-3/2}.
\]
where the prime denotes the derivative with respect to $s$. Parameterizing $\gamma_2$ by the angle $\phi$, we obtain Eq. (3).

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