Generalized Chaplygin’s Transformation and Explicit Integration of a System with a Spherical Support

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Received July 27, 2011; accepted November 19, 2011

Abstract—We discuss explicit integration and bifurcation analysis of two non-holonomic problems. One of them is the Chaplygin’s problem on no-slip rolling of a balanced dynamically non-symmetric ball on a horizontal plane. The other, first posed by Yu. N. Fedorov, deals with the motion of a rigid body in a spherical support. For Chaplygin’s problem we consider in detail the transformation that Chaplygin used to integrate the equations when the constant of areas is zero. We revisit Chaplygin’s approach to clarify the geometry of this very important transformation, because in the original paper the transformation looks a cumbersome collection of highly non-transparent analytic manipulations. Understanding its geometry seriously facilitate the extension of the transformation to the case of a rigid body in a spherical support – the problem where almost no progress has been made since Yu. N. Fedorov posed it in 1988. In this paper we show that extending the transformation to the case of a spherical support allows us to integrate the equations of motion explicitly in terms of quadratures, detect mostly remarkable critical trajectories and study their stability, and perform an exhaustive qualitative analysis of motion. Some of the results may find their application in various technical devices and robot design. We also show that adding a gyrostat with constant angular momentum to the spherical-support system does not affect its integrability.

MSC2010 numbers: 37J60, 37J35, 70E18, 70F25, 70H45
DOI: 10.1134/S1560354712020062
Keywords: nonholonomic mechanics, spherical support, Chaplygin ball, explicit integration, isomorphism, bifurcation analysis

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INTRODUCTION

In this paper we consider the problem of explicit integration and bifurcation analysis for two non-holonomic systems:

— the Chaplygin problem on no-slip rolling of a balanced dynamically non-symmetric ball on a horizontal plane,

— the problem of motion of a rigid body in a spherical support.

Mechanisms with rolling balls as part of their construction have recently received considerable interest due to possible applications in robotics and means of transport. Thus, for example, in [21] a ball model of a Mars rover is considered. In addition, more complicated machines whose design involves sophisticated arrangements of balls, which interact via the non-holonomic constraints, are treated

1) kinematically (various sorts of manipulators [6]);

2) dynamically (e.g., in the design of a Ballbot [14, 15, 17]).

Ballbots use a large ball as core of its propulsion system. This ball is driven by attached wheels (either normal wheels or omni-wheels or spherical wheels). It is interesting to note that odd-looking vehicles of the Ballbot type offer good maneuverability and are finding increasing application in various devices like Gyrover, a single-wheel robot [22]), and even in some models of vacuum cleaners [23].

The study of the dynamics of such systems is therefore of great interest. Although robotics is, as a rule, concerned primarily with control problems, the dynamics of a free system is also very important. In particular, singular stable solutions of a free system can serve as a basis for constructing relevant controls, whereas unstable trajectories should be stabilized prior to their use.

In 1903 S. A. Chaplygin integrated in terms of quadratures the equations in the ball rolling problem [5]. Since then there has appeared a large body of literature dealing with this problem. So the dynamics of this system (except for the behavior of the point of contact) has been profoundly explored. (See, for example, [7, 12, 18] and references therein; in A. A. Kilin [12], the computation from [5] is redone in terms of modern notation; an attempt at interpreting Chaplygin’s transformation was made by J. J. Duistermaat [7].) In 1988 Yu. N. Fedorov proposed the problem of rigid-body dynamics in a spherical support [9]. Apart from pointing out that this system is integrable [9], it has not been dealt with in literature. This is due to two reasons. On the one hand, the problem is not so widely known, on the other hand, it is much more complicated than the Chaplygin problem and, therefore, no general approaches have been developed to integrating such systems. In this paper we solve this problem by generalizing the transformation that was applied by S. A. Chaplygin to the integration of the ball rolling problem at a non-zero constant of areas. This highly nontrivial transformation testifies S. A. Chaplygin’s an extraordinary analytical skill, but at first sight it looks a discouraging chain of bulky and non-obvious formula manipulations. Here
we hope to elucidate the geometrical meaning of the transformation using trajectory isomorphism between two systems of the same type at different levels of the energy integral. The generalization of this transformation to the problem of a rigid body in a spherical support enables one to explicitly integrate the equations of motion in terms of quadratures, to find critical periodic solutions and perform the stability analysis.

In addition, we also show that placing a gyrostat within a body in a spherical support does not affect the system’s integrability. So this can be righteously titled a new integrable non-holonomic system.

1. CHAPLYGIN’S TRANSFORMATION IN THE PROBLEM OF A BALL’S ROLLING MOTION ON A PLANE

1.1. Equations of Motion, First Integrals, Invariant Measure

Let a balanced dynamically non-symmetric rigid body with a spherical surface (Chaplygin’s ball) roll without slipping on a fixed horizontal plane. We choose a movable reference frame related to the principal axes of inertia of the ball and denote by $\omega$, $m$, $I = \text{diag}(I_1, I_2, I_3)$, $b$ the angular velocity, mass, tensor of inertia and the radius of the ball, respectively. The angular momentum of the ball relative to the point of contact $Q$ is

$$M = \mathbf{I}\omega + \mathcal{D}\gamma \times (\omega \times \gamma), \quad \mathcal{D} = mb^2,$$

where $\gamma$ is the normal vector to the plane (see Fig. 1).

The phase space of the system is the manifold $T(SO(3) \times \mathbb{R}^2)$. As is well known, the equations governing the evolution of the vectors $M$ and $\gamma$ decouple and form a closed system

$$\dot{M} = M \times \omega, \quad \dot{\gamma} = \gamma \times \omega,$$

where, by virtue (1), the angular velocity is expressed through the angular momentum as

$$\omega = A(M + Z\gamma),$$

$$A = (I + DE)^{-1}, \quad Z = \frac{(AM, \gamma)}{D - (\gamma, A\gamma)}.$$

Thus, in this case Eqs. (2) and (3) define a four-parameter family of systems. It is convenient to parameterize the family by the eigenvalues of the matrix $A$, denoted by $D$ and $A_1, A_2, A_3$.

The system (2) admits an invariant measure $\rho dM d\gamma$ with density

$$\rho = \left(D^{-1} - (\gamma, A\gamma)\right)^{-1/2}$$

and admits four first integrals:

$$F_0 = (\gamma, \gamma) = 1, \quad F_1 = (M, M), \quad F_2 = (M, \gamma),$$

$$H = \frac{1}{2}(M, \omega) = \frac{1}{2}(M, AM) + \frac{1}{2}Z(AM, \gamma).$$

The trajectory of the point of contact of the ball and the plane can be found from the equations

$$\dot{x} = b(\omega, \beta), \quad \dot{y} = -b(\omega, \alpha),$$

where $\alpha$ and $\beta$ are unit vectors fixed in space, parallel to the plane of rolling and referred to the movable axes. In the case where $M$ and $\gamma$ are not parallel, one can choose

$$\alpha = \frac{M \times \gamma}{\sqrt{(M, M) - (M, \gamma)^2}}, \quad \beta = \frac{M - (M, \gamma)\gamma}{\sqrt{(M, M) - (M, \gamma)^2}}.$$
1.2. Explicit Integration at the Zero Constant of Areas

In the Chaplygin ball-rolling problem we set

\[ M_\gamma = (M, \gamma) = (I\omega, \gamma) = 0 \]  

and reduce the equations of motion to quadratures at a common level of the first integrals (5). To do this, we solve from the relation (8) and the equation of motion

\[ \dot{\gamma} = \gamma \times \omega, \]  

for the angular velocity \( \omega \) and the momentum \( M \), thus expressing them in terms of the vectors \( \gamma \) and \( \dot{\gamma} \). Taking the cross-product of both sides of Eq. (9) with \( I \gamma \) and using (8) and (1), we obtain

\[ \omega = k\dot{\gamma} \times I\gamma, \quad M = kI(\dot{\gamma} \times I\gamma) - D\dot{\gamma} \times \gamma, \]  

\[ k^{-1} = (\gamma, I\gamma). \]  

On the sphere \( \gamma^2 = 1 \) we determine the sphero-conical coordinates \( q_1 \) and \( q_2 \) as the roots of the equation

\[ \sum_{i=1}^{3} \frac{\gamma_i^2}{a_i - x} = 0, \]

where \( a_i = \frac{D}{l_i}, \quad i = 1, 2, 3. \) In this case

\[ \gamma_1^2 = \frac{(a_1 - q_1)(a_1 - q_2)}{(a_1 - a_2)(a_1 - a_3)}, \quad \gamma_2^2 = \frac{(a_2 - q_1)(a_2 - q_2)}{(a_2 - a_1)(a_2 - a_3)}, \quad \gamma_3^2 = \frac{(a_3 - q_1)(a_3 - q_2)}{(a_3 - a_1)(a_3 - a_2)}. \]

We differentiate these relations with respect to time, then use (10) and represent the quadratic integrals of motion \( H \) and \( F_1 \) as functions of the generalized coordinates and velocities \( q_i, \dot{q}_i, \) \( i = 1, 2 \) as follows:

\[ H = \frac{D(q_1 - q_2)}{2} \Phi \left[ \frac{\dot{q}_1^2}{q_2 \varphi(q_1)} - \frac{\dot{q}_2^2}{q_1 \varphi(q_2)} \right], \quad F_1 = \frac{D^2(q_1 - q_2)}{2} \Phi \left[ \frac{(1 + q_2)\dot{q}_1^2}{q_2^2 \varphi(q_1)} - \frac{(1 + q_1)\dot{q}_2^2}{q_1^2 \varphi(q_2)} \right], \]

\[ \Phi = \frac{(1 + q_1)(1 + q_2)}{4}, \quad \varphi(x) = (1 + x)(a_1 - x)(a_2 - x)(a_3 - x). \]

At the fixed common level of the first integrals \( 2H = \mathcal{E}, \) \( F_1 = \mathcal{C} \), where \( \mathcal{E} \) and \( \mathcal{C} \) are some constants, we obtain the quadratures

\[ \dot{q}_1^2 = \frac{q_2^2 \varphi(q_1)(Cq_1 - \mathcal{E}D(1 + q_1))}{D^2(q_1 - q_2)}, \quad \dot{q}_2^2 = \frac{q_1^2 \varphi(q_2)(Cq_2 - \mathcal{E}D(1 + q_2))}{D^2(q_1 - q_2)}. \]  

**Remark.** Separation variables of the system (2) on \( (M, \gamma) = 0 \) can be found via a simple procedure first offered by Eisenhart [8]. Choose arbitrarily local variables \( x = (x_1, x_2) \) on the sphere \( \gamma^2 = 1; \) according to (10), the first integrals \( H \) and \( F_1 \) are homogeneous quadratic forms in the velocities \( \dot{x} = (\dot{x}_1, \dot{x}_2) \)

\[ H = (\dot{x}, B_1 \dot{x}), \quad F_1 = (\dot{x}, B_2 \dot{x}), \]

where \( B_1 \) and \( B_2 \) are \( 2 \times 2 \)-matrices depending on the \( x \)-coordinates. Then the eigenvalues \( \lambda_1(x) \) and \( \lambda_2(x) \) of the matrix \( B_1B_2^{-1} \) are separation variables of the system. As a rule, to simplify the formulae, it is more convenient to take the functions of these variables \( q_1(\lambda_1), \) \( q_2(\lambda_2). \)
1.3. Linear Trajectory Isomorphism

It follows from Eqs. (2) that the vectors $\mathbf{M}$ and $\gamma$ are constant in a fixed frame of reference. Hence, their linear combinations (with constant coefficients) are also constant, while in a movable frame of reference their evolution is governed by the same equations:

$$
\ddot{\mathbf{M}} = m_1 \mathbf{M} + m_2 \gamma, \quad \ddot{\gamma} = c_1 \mathbf{M} + c_2 \gamma,
$$

$$
\dot{\mathbf{M}} = \mathbf{M} \times \omega, \quad \dot{\gamma} = \gamma \times \omega.
$$

(12)

For the transformed vectors $\tilde{\mathbf{M}}$ and $\tilde{\gamma}$ we determine the transformed angular velocity $\tilde{\omega}$ by the relation coinciding with (3):

$$
\tilde{\omega} = \mathbf{A}(\tilde{\mathbf{M}} + \tilde{Z}\tilde{\gamma}), \quad \tilde{Z} = \frac{(\mathbf{A}\tilde{\mathbf{M}}, \tilde{\gamma})}{D^{-1} - (\mathbf{A}\tilde{\gamma}, \tilde{\gamma})}.
$$

(13)

**Theorem 1.** If at a fixed level of the energy integral $2H = \mathcal{E}$ the transformation coefficients (12) satisfy the relations

$$
R_1 = m_2 D^{-1} + \mathcal{E}c_1 \Delta = 0, \quad R_2 = m_1 D^{-1} - c_2 \Delta D^{-1} = 0,
$$

$$
\Delta = m_1 c_2 - m_2 c_1,
$$

(14)

then the vectors $\omega$ and $\tilde{\omega}$ are proportional:

$$
\omega = \lambda \tilde{\omega},
$$

(15)

where the coefficient $\lambda$ is some function of phase variables $\mathbf{M}$ and $\gamma$.

**Proof.** Denote the denominator and the numerator of the quantity $Z$ as follows:

$$
X = D^{-1} - (\gamma, \mathbf{A}\gamma), \quad Y = (\mathbf{A}\mathbf{M}, \gamma).
$$

(16)

Then, transforming (12) and using the relations $(\mathbf{A}\mathbf{M}, \mathbf{M}) = \mathcal{E} - X^{-1}Y^2$, $(\gamma, \mathbf{A}\gamma) = D^{-1} - X$, we obtain

$$
\tilde{X} = D^{-1} - c_1^2(\mathcal{E} - X^{-1}Y^2) - c_2^2(D^{-1} - X) - 2c_1c_2 Y,
$$

$$
\tilde{Y} = m_1 c_1 (\mathcal{E} - X^{-1}Y^2) + m_2 c_2 (D^{-1} - X) + (m_1 c_2 + m_2 c_1) Y.
$$

(17)

We represent the proportionality condition for the vectors $\omega$ and $\tilde{\omega}$ in the form

$$
\Delta_{ij} = \omega_i \tilde{\omega}_j - \omega_j \tilde{\omega}_i = 0, \quad i \neq j.
$$

Using (13) and (17) we find

$$
\Delta_{ij} = A_i A_j (M_i \gamma_j - M_j \gamma_i) \Phi(X, Y),
$$

$$
\Phi(X, Y) = \frac{R_2 Y - R_1 X}{(c_1^2 + D^{-1} c_2^2 - D^{-1})X - (c_2 X - c_1 Y)^2},
$$

where $A_i = (I_i + D)^{-1}$ are the elements of the matrix $\mathbf{A}$. From this we infer that the proportionality condition is satisfied for $\Phi(X, Y) = 0$, i.e. for arbitrary $X$ and $Y$ it is determined by the relation (14). \hfill \Box

It is also straightforward to see that the proportionality coefficient $\lambda$ is given by

$$
\lambda = m_1 + c_1 \tilde{Z}.
$$

(18)

According to this theorem, in terms of new variables the equations of motion are recast as

$$
\ddot{\tilde{\mathbf{M}}} = \lambda \tilde{\mathbf{M}} \times \tilde{\omega}, \quad \ddot{\tilde{\gamma}} = \lambda \tilde{\gamma} \times \tilde{\omega},
$$

$$
\tilde{\omega} = \mathbf{A}(\tilde{\mathbf{M}} + \tilde{Z}\tilde{\gamma}), \quad \tilde{Z} = \frac{(\mathbf{A}\tilde{\mathbf{M}}, \tilde{\gamma})}{D^{-1} - (\tilde{\gamma}, \mathbf{A}\tilde{\gamma})},
$$

(19)

i.e. they differ from the original equations (2) only in that they have an additional coefficient $\lambda$, which can be eliminated by means of time change.
We note that although, according to (3), the quantity $\mathcal{D}$ also appears in the matrix $A$, our definition of $\tilde{\omega}$ implies that $A$ does not change, whereas the denominator $\tilde{Z}$ contains not $\mathcal{D}$ but $\tilde{D}$, i.e., in this case the moments of inertia of the ball also change, such that $A$ itself remains unchanged! Thus, this transformation is no symmetry of the system but defines the trajectory isomorphism with another system of the same type.

The level of the energy integral $(\tilde{M}, \tilde{\omega}) = \tilde{E}$ of the system (19) is related to the level of the energy of the initial system by

$$\tilde{E} = \mathcal{E}m_1^2 + \mathcal{D}^{-1}m_2^2.$$  \hspace{1cm} (20)

It is convenient to pass from Eqs. (14) to the equivalent system of relations (using the linear combinations $R_1$ and $R_2$):

$$\mathcal{E}m_1c_1 + \mathcal{D}^{-1}m_2c_2 = 0, \quad \tilde{D}^{-1} = \mathcal{E}c_1^2 + \mathcal{D}^{-1}c_2^2.$$  \hspace{1cm} (21)

Remark 1. The relations (20) and (21) have a remarkable geometrical interpretation. For two-dimensional vectors we determine the metric of the form

$$\langle x, y \rangle = \mathcal{E}x_1y_1 + \mathcal{D}^{-1}x_2y_2,$$

in which case the lines of the Chaplygin transformation matrix

$$Q_\mathcal{E} = \begin{vmatrix} m_1 & m_2 \\ c_1 & c_2 \end{vmatrix}$$

turn out to be orthogonal with respect to the metric of the form

$$\langle m, c \rangle = 0,$$

where $m = (m_1, m_2)$ and $c = (c_1, c_2)$. Moreover, the new constant $\tilde{D}$ and the value of the energy integral $\tilde{E} = (\tilde{M}, \tilde{\omega})$ are determined by

$$\tilde{D}^{-1} = \langle c, c \rangle, \quad \tilde{E} = \langle m, m \rangle.$$

1.4. Reduction to a Zero Constant of Areas (Chaplygin’s Transformation)

Without loss of generality one can set

$$F_0 = (\gamma, \gamma) = 1, \quad F_1 = (M, M) = C, \quad F_2 = (M, \gamma) = M_\gamma,$$

with $|M_\gamma| \leq \sqrt{C}$.

We parametrize the relations (21) as follows:

$$m_1 = \eta \cos \theta, \quad m_2 = -\eta k \sin \theta, \quad c_1 = \delta \sin \theta, \quad c_2 = \delta k \cos \theta,$$

$$k^2 = \mathcal{E}D, \quad \delta^2 = D\tilde{D}^{-1}.$$

In this case we obtain a three-parameter family of linear transformations (at a fixed energy level), which is represented as:

$$\tilde{M} = \eta(\cos \theta M - k \sin \theta \gamma), \quad \tilde{\gamma} = \delta(\sin \theta M + k \cos \theta \gamma).$$  \hspace{1cm} (22)

The parameters $\theta$, $\eta$ and $\delta$ can be chosen such that the vectors $\tilde{M}$ and $\tilde{\gamma}$ become orthonormalized.

The orthogonality condition $(\tilde{M}, \tilde{\gamma}) = 0$ leads to the equation determining the angle $\theta$ as follows:

$$\tan 2\theta = -\frac{2kM_\gamma}{C - k^2}.$$  \hspace{1cm} (23)

Two solutions of this equation determine the angles $\theta_1$ and $\theta_2$ that differ by $\frac{\pi}{2}$, i.e., the vectors $\tilde{M}$ and $\tilde{\gamma}$ are defined with an accuracy up to rearrangement. Since the normality conditions are $(\tilde{M}, \tilde{M}) = \tilde{C}$, $(\tilde{\gamma}, \tilde{\gamma}) = 1$, we find the parameters $\eta$ and $\delta$:

$$\eta = \sqrt{C}(C \cos^2 \theta + k^2 \sin^2 \theta - 2kM_\gamma \sin \theta \cos \theta)^{-\frac{1}{2}}, \quad \delta = (C \sin^2 \theta + k^2 \cos^2 \theta + 2kM_\gamma \cos \theta \sin \theta)^{-\frac{1}{2}}.$$
Remark. The transformation (12), which is used in Theorem 1 and reduces the non-zero constant of areas to a zero constant in the Chaplygin ball, thereby establishing the isomorphism between two different systems, is linear in phase variables but with coefficients depending on the constants of first integrals. The analogous transformation also occurs in other systems. For example, Bobenko’s transformation, which relates the Clebsch and Steklov case on different Lie algebras (see [2]). In Fedorov [10] this transformation was used to integrate one non-holonomic system.

1.5. Bifurcation Analysis and Stability of Periodic Solutions

The case $(M, \gamma) = 0$. In this case the quadratures (11) easily allow determination of the critical trajectories of the system and construction of a bifurcation diagram. The bifurcation values of the integrals $\mathcal{C}$ and $\mathcal{E}$ are determined from the multiplicity condition for the roots of the polynomial

$$R(x) = \varphi(x)((\mathcal{C} - \mathcal{E}D)x - ED).$$

Hence we find three bifurcation curves on the plane of the first integrals (see Fig. 2)

$$\mathcal{E} = A_i \mathcal{C}, \quad i = 1, 2, 3. \quad (24)$$

It is straightforward to show that to these curves there correspond the periodic solutions of the original equations (2)

$$M = \pm \sqrt{\mathcal{C}} e_i, \quad \gamma = \cos \varphi e_j + \sin \varphi e_k, \quad \varphi = \pm A_i \sqrt{\mathcal{C}} t, \quad (25)$$

where $e_i$ are the vectors of the principal dynamical axes, $i, j, k$ is a cyclic rearrangement of the numbers 1, 2, 3. These solutions govern a uniform rolling straight-line motion of a ball where the principal axis vector $e_i$ is parallel to the plane, while the angular velocity vector is parallel to the vector $M$ and is equal to $\omega = \pm A_i \sqrt{\mathcal{C}} e_i$. The relations (25) govern six families (taking into account the sign that determines the direction of rotation) of the critical trajectories of the system, i.e. to each straight line (24) in Fig. 2 there correspond two families.

Fig. 2. Bifurcation diagram in the case $(M, \gamma) = 0$ (we assume that $A_1 < A_2 < A_3$).

Using the results of [1], it is straightforward to construct a bifurcation complex of this system (Fig. 2b). Since the periodical solutions of the integrable conformally Hamiltonian system are stable if and only if they lie at the edge of the bifurcation complex, we conclude in this case that there exist four families of stable periodic solutions that are defined by the curves $\sigma_1$ and $\sigma_3$. 
The case \((\mathbf{M}, \gamma) \neq 0\). Since the transformation (22) determines the trajectory equivalence (of various systems), we make use of them to find critical trajectories for \(\mathbf{M}_\gamma \neq 0\).

Set
\[
(\tilde{\mathbf{M}}, \tilde{\gamma}) = 0, \quad (\tilde{\mathbf{M}}, \tilde{\mathbf{M}}) = \tilde{\mathbf{C}}, \quad (\gamma, \tilde{\gamma}) = 1
\]
and, using (22), we find
\[
\tilde{\mathbf{E}} = k^2\eta^2D^{-1}, \quad \tilde{\mathbf{C}} = \frac{\eta^2}{2}(\mathbf{C} + k^2 + (\mathbf{C} - k^2)\cos 2\theta - 2k\mathbf{M}_\gamma \sin 2\theta).
\]
Using the condition for criticality of the trajectories (at \((\tilde{\mathbf{M}}, \tilde{\gamma}) = 0\))
\[
\tilde{\mathbf{E}} = A_i\tilde{\mathbf{C}}, \quad i = 1, 2, 3,
\]
and the relation (23), after straightforward simplifications we derive an equation that defines bifurcation curves in the general case, in the form
\[
(k^2 - \mathcal{D}A_i\mathbf{C})(1 - \mathcal{D}A_i) = \mathcal{D}^2A_i^2M_\gamma^2.
\]
Finally we find the bifurcation curves corresponding to the straight lines (26):
\[
\sigma_i: \mathbf{E} = A_i\mathbf{C} + \frac{A_i^2}{D^{-1} - A_i}M_\gamma^2, \quad i = 1, 2, 3.
\]

In order to obtain the corresponding critical solutions we apply the backward transformation \(\mathbf{Q}_\mathbf{E}^{-1}\) to the critical solutions (25):
\[
\mathbf{M} = M_i\mathbf{e}_i + M_\perp(\cos \varphi \mathbf{e}_j + \sin \varphi \mathbf{e}_k), \quad \gamma = \gamma_i\mathbf{e}_j + \gamma_\perp(\cos \varphi \mathbf{e}_j + \sin \varphi \mathbf{e}_k),
\]
\[
M_i^2 = \cos^2 \theta \frac{\tilde{\mathbf{C}}}{\eta^2}, \quad \gamma_i^2 = \sin^2 \theta \frac{\tilde{\mathbf{C}}}{\eta^2k^2}, \quad M_\perp^2 = \mathbf{C} - M_i^2, \quad \gamma_\perp = 1 - \gamma_i^2.
\]

By virtue of the basic property of the transformation \(\mathbf{Q}_\mathbf{E}\) the angular velocity is proportional to the initial velocity: using (25) and (18) we find
\[
\omega = \mathbf{A}(\mathbf{M} + Z\gamma) = \Lambda A_i\mathbf{e}_i, \quad \Lambda = \pm \eta \cos \theta \sqrt{\tilde{\mathbf{C}}}.
\]
Hence, for these critical solutions the equation holds
\[
\mathbf{M} + Z\gamma = \Lambda \mathbf{e}_i.
\]

In order to find the dependence of the parameter of this solution through the values of the first integrals of the system, we successively perform scalar multiplication of this equation by the vectors \(\mathbf{M}, \gamma, \mathbf{e}_i\) and \(\mathbf{A}\gamma\) and derive the equations
\[
\mathbf{C} + ZM_\gamma = \mathbf{\Lambda M}_i, \quad M_\gamma + Z = \Lambda \gamma_i, \quad M_i + Z\gamma_i = \Lambda, \quad \mathcal{D}^{-1}Z = \Lambda A_i\gamma_i,
\]
where we make use of the identity \((\mathbf{A}\mathbf{M}, \gamma) + Z(\gamma, \mathbf{A}\gamma) = \mathcal{D}^{-1}Z\). This yields
\[
\mathbf{\Lambda M}_i = \mathbf{C} + \frac{A_i}{\mathcal{D}^{-1} - A_i}M_\gamma^2, \quad \Lambda \gamma_i = \frac{\mathcal{D}^{-1}}{\mathcal{D}^{-1} - A_i}M_\gamma,
\]
\[
\Lambda^2 = \mathbf{C} + \frac{(2\mathcal{D}^{-1} - A_i)A_i}{(\mathcal{D}^{-1} - A_i)^2}M_\gamma^2, \quad Z = \frac{A_i}{\mathcal{D}^{-1} - A_i}M_\gamma.
\]
These solutions also govern the rolling motion of a ball along a straight line, but in this case the angular velocity vector \(\omega\) directed along the principal axis \(\mathbf{e}_i\) is constant in fixed axes and forms with the vertical an angle \(\psi\), which is determined by
\[
\cos \psi = (\mathbf{e}_i, \gamma) = \gamma_i.
\]
To each value of \(M_\gamma\) there correspond two critical trajectories with different signs in the expression for \(\Lambda\).
In addition to these critical solutions (when the vectors $M$ and $\gamma$ are not collinear), for $M_\gamma \neq 0$ the system admits critical solutions for which the equations hold

$$M = \lambda \gamma, \quad \lambda = M_\gamma = \text{const.}$$

To these solutions there corresponds a bifurcation curve (more precisely, a surface) given by

$$C = M_\gamma^2.$$

The dynamics of the vector $\gamma$ is governed by the Euler equation for the motion of a free rigid body with a fixed point

$$\dot{\gamma} = \gamma \times \tilde{A} \gamma, \quad \tilde{A} = M_\gamma \left(1 + \frac{ED}{C}\right) A,$$

while the trajectory lies at the intersection of the sphere and the ellipsoid

$$(\gamma, \gamma) = 1, \quad (\gamma, A \gamma) = \frac{E}{C + ED}.$$

Obviously, this intersection defines on the Poisson sphere $\gamma^2 = 1$ two symmetric curves; consequently, to each value of $M_\gamma$ there correspond two different critical trajectories.

Let us construct a bifurcation diagram on the plane of the first integrals $E = \frac{ED}{C}, g = \frac{M_\gamma}{\sqrt{C}}$ (this is equivalent to reducing the value of the integral $(M, M)$ to one by performing the time substitution $\sqrt{C} \, dt \rightarrow dt$). The diagram is given by five curves (see Fig. 3a)

$$\sigma_i: E = b_i + \frac{b_i^2 - g^2}{1 - b_i}, \quad b_i = DA_i = \frac{D}{I_i + D}, \quad i = 1, 2, 3,$$

$$\sigma_4: g = 1, \quad \sigma_5: g = -1,$$

with $0 < b_i < 1$ being constant.

As in the case $(M, \gamma) = 0$, we construct the bifurcation complex using the theorem that all stable periodic solutions lie at the edge of the bifurcation complex. The corresponding bifurcation complex is shown in Fig. 3b. Taking into account that to each point of all bifurcation curves there correspond two critical trajectories, we conclude that in this case there exist 8 families of stable periodic solutions given by the curves $\sigma_1$, $\sigma_3$, $\sigma_4$ and $\sigma_5$ and lying on four leaves of the bifurcation complex. These leaves stick together along the straight line $\sigma_2$ on which lie unstable critical trajectories and unstable invariant manifolds (separatrices) that connect them. From each family corresponding to the curves $\sigma_4$ and $\sigma_5$ we need to eliminate one point that lies on the intersection with the curve $\sigma_2$ and determines unstable equilibrium configurations.

2. A RIGID BODY IN A SPHERICAL SUPPORT

2.1. Equations of Motion, First Integrals, Invariant Measure

The problem of motion of a body in a spherical support, which was proposed by Yu. N. Fedorov in [9], is an integrable generalization of the Chaplygin ball rolling problem. This system consists of a body rigidly enclosed in a spherical shell, so that the center of mass of both bodies coincides with the geometrical center of the shell, and of an arbitrary number of dynamically symmetric balls in contact with the shell (Fig. 4) without slipping at the points of contact. The centers of the balls and the shell are fixed in space; in what follows we shall, for brevity, refer to the body within the shell as the central ball.

The no-slip condition is that the velocities of the points of contact of different balls coincide. They are represented by vector equations of the form

$$f^{(k)} = \omega \times n^{(k)} + r_k \omega^{(k)} \times n^{(k)} = 0,$$

$$k = 1, \ldots, N,$$ (27)
where $N$ is the number of balls adjoining the central one, $\omega$ and $\omega^{(k)}$ are the angular velocities of the corresponding balls, $\mathbf{n}^{(k)}$ are the space-fixed unit vectors directed from the center of the shell to the points of contact and $r_k = \frac{R_k}{R_0}$ is the ratio of radii of the balls (Fig. 4).

We choose a movable frame of reference related to the principal axes of inertia of the central ball. The equations of motion of the balls can be derived from the Ferrers equations with undetermined multipliers:

$$(\frac{\partial T}{\partial \omega}) = \frac{\partial T}{\partial \omega} \times \omega + \sum_{k,i} \lambda^{(k)}_i \frac{\partial f^{(k)}_i}{\partial \omega},$$

$$\left(\frac{\partial T}{\partial \omega^{(k)}}\right) = \left(\frac{\partial T}{\partial \omega^{(k)}}\right) \times \omega + \lambda^{(k)} \frac{\partial f^{(k)}_i}{\partial \omega^{(k)}},$$

where $T$ is the kinetic energy of a free system

$$T = \frac{1}{2}(\omega, \mathbf{I}\omega) + \frac{1}{2} \sum_k \mu_k (\omega^{(k)}, \omega^{(k)}),$$

(28)

here $\mathbf{I} = \text{diag}(I_1, I_2, I_3)$ is the tensor of inertia of the central ball, $\mu_k$ are the moments of inertia of the adjoining dynamically symmetric balls and $\lambda^{(k)}_i$ are undetermined multipliers, i.e. constraint reactions.

Supplementing these equations with the kinematic equations governing the evolution of the coordinates of fixed vectors $\mathbf{n}^{(k)}$ in movable axes, we obtain

$$\mathbf{I}\dot{\omega} = \mathbf{I}\omega \times \omega + \sum_k \mathbf{n}^{(k)} \times \mathbf{N}^{(k)}, \quad \mu_k \dot{\omega}^{(k)} = \mu_k \omega^{(k)} \times \omega + r_k \mathbf{n}^{(k)} \times \mathbf{N}^{(k)},$$

(29)

where $\mathbf{N}^{(k)} = (\lambda^{(k)}_1, \lambda^{(k)}_2, \lambda^{(k)}_3)$.

Using these equations, it is straightforward to show that the projections of angular velocities $\omega^{(k)}$ onto the vectors $\mathbf{n}^{(k)}$ remain unchanged, i.e. they are first integrals

$$\Lambda_k = (\omega^{(k)}, \mathbf{n}^{(k)}) = \text{const.}$$

(30)

Using these integrals we eliminate the constraint reactions $\mathbf{N}^{(k)}$ and obtain a closed system describing the dynamics of the central ball. To this end we multiply the constraints (27) vectorially.
by $n^{(k)}$ and obtain
\[ \omega^{(k)} = -\frac{1}{r_k} (\omega - (\omega, n^{(k)}) n^{(k)}) + \Lambda_k n^{(k)}. \]
Differentiating this relation and substituting into (29), we finally find
\[ I \dot{\omega} + \sum_k \frac{\mu_k}{r_k} (\dot{(\omega, n^{(k)}), n^{(k)}}) = I \omega \times \omega, \]
where $E$ is represented in the form (for comparison, see (13))
\[ \omega^{(k)} = \frac{1}{r_k} (\omega - (\omega, n^{(k)}) n^{(k)}) + \Lambda_k n^{(k)}. \]

We define the vector
\[ M = \left( I + \sum_k \frac{\mu_k}{r_k} E \right) \omega - \sum_k \frac{\mu_k}{r_k} (\omega, n^{(k)}) n^{(k)} = \left( I + \sum_k \frac{\mu_k}{r_k} E - \sum_k \frac{\mu_k}{r_k} n^{(k)} \otimes n^{(k)} \right) \omega, \]
where $E$ is the unit matrix, $a \otimes b = \|a\|b$; using (31) we find that the evolution of $M$ is given by
\[ \dot{M} = M \times \omega, \]
consequently, $M$ is fixed in space.

In a fixed frame of reference the tensor
\[ D = \sum_k \frac{\mu_k}{r_k} n^{(k)} \otimes n^{(k)} \]
is represented by a symmetric matrix with constant coefficients. Let $\alpha$, $\beta$ and $\gamma$ be the basis of eigenvectors (fixed in space) of this matrix, so that
\[ D = D^{(0)}_\alpha \alpha \otimes \alpha + D^{(0)}_\beta \beta \otimes \beta + D^{(0)}_\gamma \gamma \otimes \gamma = D^{(0)}_\gamma E + (D^{(0)}_\alpha - D^{(0)}_\gamma) \alpha \otimes \alpha + (D^{(0)}_\beta - D^{(0)}_\gamma) \beta \otimes \beta, \]
where $D^{(0)}_\alpha$, $D^{(0)}_\beta$ and $D^{(0)}_\gamma$ are the eigenvalues of the tensor $D$.

Finally we represent the equations of motion of the central ball in movable axes as a closed system of nine equations of the form
\[ \dot{M} = M \times \omega, \quad \dot{\alpha} = \alpha \times \omega, \quad \dot{\beta} = \beta \times \omega, \]
where the angular velocity $\omega$ is expressed in terms of the vectors $M$, $\alpha$ and $\beta$ using the linear relations
\[ M = (J - D_\alpha \alpha \otimes \alpha - D_\beta \beta \otimes \beta) \omega, \]
\[ J = I + \sum_k \frac{\mu_k}{r_k} E - D^{(0)}_\gamma E, \quad D_\alpha = D^{(0)}_\alpha - D^{(0)}_\gamma, \quad D_\beta = D^{(0)}_\beta - D^{(0)}_\gamma. \]

In order to close Eqs. (32), we need to express the angular velocity from (33); it can be shown that $\omega$ is represented in the form (for comparison, see (13))
\[ \omega = A (M + Z_\alpha \alpha + Z_\beta \beta), \quad A = J^{-1}, \]
\[ \begin{pmatrix} Z_\alpha \\ Z_\beta \end{pmatrix} = \begin{pmatrix} D^{-1}_\alpha - (\alpha, A \alpha) \\ -(\alpha, A \beta) \end{pmatrix}^{-1} \begin{pmatrix} Y_\alpha \\ Y_\beta \end{pmatrix}, \]
where $Y_\alpha = (A M, \alpha)$, $Y_\beta = (A M, \beta)$.

Eqs. (32) admit the following (obvious) first integrals:

- geometric integrals
\[ \alpha^2 = 1, \quad \beta^2 = 1, \quad (\alpha, \beta) = 0, \]
— projections of the angular momentum onto the fixed axes
\[(M, \alpha) = \text{const}, \quad (M, \beta) = \text{const}, \quad (M, \gamma) = (M, \alpha \times \beta) = \text{const}, \quad (36)\]

— energy
\[H = \frac{1}{2}(M, \omega). \quad (37)\]

The integrals (36) are analogs of the constant of areas \(F_2\) in the Chaplygin problem (5).

**Remark 2.** The system (32) also admits an integral of the squared momentum
\[(M, M) = (M, \alpha)^2 + (M, \beta)^2 + (M, \alpha \times \beta)^2 = \text{const.} \quad (39)\]

In addition, Eqs. (32) also admit an invariant measure \(\rho dM d\alpha d\beta\) with the density
\[\rho = \left[\det(J - D_\alpha \alpha \otimes \alpha - D_\beta \beta \otimes \beta)\right]^{-\frac{1}{2}}. \quad (38)\]
Hence, by the Euler–Jacobi theorem this system is integrable.

### 2.2. A Support with Rubber Balls

As is well known, robotics deals not only with systems with non-slip constraints (27) but also with systems where both slipping and twisting is prohibited at the point of contact of bodies. In this case, for a spherical support a full system of constraints in the notation of Section 2.1 is written as
\[f^{(k)} = \omega \times n^{(k)} + r_k \omega (k) \times n^{(k)} = 0, \quad g^{(k)} = (\omega - \omega^{(k)}, n^{(k)}) = 0. \quad (39)\]

The model of rolling without slipping and twisting, for which the contacting surfaces satisfy the constraints (39), is proposed in [3, 13] where the authors call it, more figuratively, the model of rolling of a rubber body. The spherical support with rubber balls was first considered in [11], where its integrability by the Euler–Jacobi theorem is shown. Here we show that the problem at hand is not new but reduces to the previous one.

Writing the Ferrers equations with kinetic energy (28) in the system of principal axes of inertia of the central ball, we obtain
\[I \omega = I \omega \times \omega + \sum_k n^{(k)} \times N^{(k)} + \sum_k \lambda_k n^{(k)}, \quad (40)\]
\[\mu_k \dot{\omega}^{(k)} = \mu_k \omega^{(k)} \times \omega + r_k \omega n^{(k)} \times N^{(k)} - \lambda_k n^{(k)}, \quad (41)\]
where \(\lambda_k\) are undetermined multipliers of the constraints \(g^{(k)} = 0\).

From the constraint equations (39) we find
\[\omega^{(k)} = -\frac{1}{r_k} \omega + \left(1 + \frac{1}{r_k}\right) (\omega, n^{(k)}) n^{(k)} \quad (41)\]
and as a consequence of this equation and the equations of motion (40) we obtain
\[(\omega, \dot{n}^{(k)}) = (\omega^{(k)}, \dot{n}^{(k)}) = 0, \quad (\omega, n^{(k)}) = (\dot{\omega}, n^{(k)}). \quad (42)\]

Using (40), (41) and (42) we express the undetermined multipliers as
\[\lambda_k = -\mu_k (\omega, n^{(k)}), \quad \mu_k = \frac{\mu_k}{r_k} \omega + \frac{\mu_k}{r_k} \left(1 + \frac{1}{r_k}\right) (\omega, n^{(k)}) n^{(k)} + \frac{\lambda_k}{r_k} n^{(k)}. \]
As in the previous case, we obtain a closed system governing the dynamics of the central ball
\[ I\dot{\omega} + \sum_k \mu_k \frac{1}{r_k^2} \omega - \sum_k \mu_k \left( \frac{1}{r_k^2} - 1 \right) \left( \omega, n^{(k)} \right) n^{(k)} = I\omega \times \omega, \]
\[ \dot{n}^{(k)} = n^{(k)} \times \omega. \]

By analogy with a usual spherical support we define the vector
\[ M = \left( I + \sum_k \frac{\mu_k}{r_k^2} E - \sum_k \mu_k \left( \frac{1}{r_k^2} - 1 \right) n^{(k)} \otimes n^{(k)} \right) \omega \]
and the symmetric tensor
\[ D' = \sum_k \mu_k \left( \frac{1}{r_k^2} - 1 \right) n^{(k)} \otimes n^{(k)}. \]

It is straightforward to show that their components are constant in fixed axes.

In absolute space we choose a basis for the eigenvectors of \( D' \):
\[ D' = D'_\alpha \alpha \otimes \alpha + D'_\beta \beta \otimes \beta + D'_\gamma \gamma \otimes \gamma = D'_\alpha E + D'_\alpha \alpha \otimes \alpha + D'_\beta \beta \otimes \beta \]
\[ D'_\alpha = D'_\alpha - D'_\gamma, \quad D'_\beta = D'_\beta - D'_\gamma; \]
then the equations of motion of the central ball in a rubber spherical support are
\[ \dot{M} = M \times \omega, \quad \dot{\alpha} = \alpha \times \omega, \quad \dot{\beta} = \beta \times \omega, \]
where
\[ M = \left( J' - D'_\alpha \alpha \otimes \alpha - D'_\beta \beta \otimes \beta \right) \omega, \]
\[ J' = I + \left( \sum_k \frac{\mu_k}{r_k^2} - D'_\gamma \right) E. \]

Thus, the problem of motion of a body in a spherical support with rubber balls is equivalent to the problem of motion of a body in a usual spherical support. (The difference arises only in the law of motion of homogeneous balls holding the central ball.)

2.3. A Gyrostat in a Spherical Support

Consider the generalization of the problem of a body in a spherical support to the case where a balanced rotor is attached to the central ball and rotates with constant velocity \( \omega_r \) about the axis fixed in a body. We shall call such a system a gyrostat in a spherical support. In this case the kinetic energy of a free system has the representation
\[ T = \frac{1}{2} (\omega, I\omega) + (K, \omega) + \frac{1}{2} \sum_k \mu_k (\omega^{(k)}, \omega^{(k)}) + \frac{1}{2} I_r \omega_r^2, \]
where \( K = I_r \omega_r e_r \) is the gyrostatic moment vector, directed along the rotor axis whose direction is determined by the body-fixed vector \( e_r, I_r \) is the moment of inertia of the rotor relative to this axis and \( I \) is the tensor of inertia of the entire system ball + rotor.

Repeating the reasoning of the previous section, one can show that the equations of motion of this system are
\[ \dot{M} = (M + K) \times \omega, \quad \dot{\alpha} = \alpha \times \omega, \quad \dot{\beta} = \beta \times \omega, \]
where \( M \) and \( \omega \) are also related by (33) and (34), i.e.
\[ \omega = A(M + Z_\alpha \alpha + Z_\beta \beta). \]
These equations admit seven independent first integrals:
— geometric integrals
\[ \alpha^2 = 1, \quad \beta^2 = 1, \quad (\alpha, \beta) = 0, \]
— projections of the generalized angular momentum onto the fixed axes
\[ (M + K, \alpha) = \text{const}, \quad (M + K, \beta) = \text{const}, \quad (M + K, \alpha \times \beta) = \text{const}, \]
— energy
\[ H = \frac{1}{2}(M + K, \omega). \]

These equations also admit an invariant measure \( \rho dM d\alpha d\beta \) with the density \( \rho(\alpha, \beta) \) given by (38).

Thus, the system is integrable by the Euler–Jacobi theorem.

Remark 3. A constant gyrostatic moment can arise even if there is no rotor, in the case where the body contains multiply connected cavities filled with an ideal fluid whose flow has a non-zero circulation [16].

2.4. A Spherical Support in the Case \( M \parallel \beta \)

Let us take the values of two “constant areas” in (32) to be zero. Without loss of generality one can set
\[ (M, \alpha) = 0, \quad (M, \gamma) = (M, \alpha \times \beta) = 0. \tag{43} \]

Then the relation holds
\[ M = sv\sqrt{C}\beta, \]
where \( (M, M) = C \) is the square of the angular momentum, \( s = \pm 1 \) depending on the mutual directions of the vectors \( M \) and \( \beta \). Using this equality, at the level of the energy integral (37) \( 2H = \mathcal{E} \) we obtain
\[ (\beta \otimes \beta)\omega = \frac{\mathcal{E}}{C}M. \]

Substitute this relation into the definition of the vector \( M \) (33) and introduce a new vector of the angular momentum
\[ \tilde{M} = \left( 1 + \frac{\mathcal{E}D_\beta}{C} \right) M = (J - D_\alpha \alpha \otimes \alpha)\omega. \]

From this equality we conclude that the following holds

**Proposition 1.** The system (33) governing the motion of a rigid body in a spherical support, subject to the constraints (43) and with fixed values of the integrals \( (M, M) = C \) and \( (M, \omega) = \mathcal{E} \), is mapped using the transformation
\[ \tilde{M} = \left( 1 + \frac{\mathcal{E}D_\beta}{C} \right) M, \quad \tilde{\gamma} = \alpha \]
into the system for the Chaplygin ball
\[ \tilde{M} = \tilde{M} \times \tilde{\omega}, \quad \tilde{\gamma} = \tilde{\gamma} \times \tilde{\omega}, \]
\[ \tilde{\omega} = \omega = A \left( \frac{(A\tilde{M}, \tilde{\gamma})}{D_\alpha^{-1} - (\tilde{\gamma}, A\tilde{\gamma})} \right), \quad A = J^{-1} \]
at fixed values of the integrals
\[ \tilde{C} = (\tilde{M}, \tilde{M}) = C \left( 1 + \frac{\mathcal{E}D_\beta}{C} \right)^2, \quad \tilde{\mathcal{E}} = (\tilde{M}, \tilde{\omega}) = \mathcal{E} \left( 1 + \frac{\mathcal{E}D_\beta}{C} \right), \quad \tilde{M}, \tilde{\gamma} = 0. \tag{44} \]
Remark. In the case where one of the “constant areas” is equal to zero, the problem of motion of a body in a spherical support also reduces to the Chaplygin ball rolling problem but at an arbitrary constant of areas. Although this is not used anywhere, we point out this transformation; for definiteness we set

\[(M, \gamma) = 0, \quad \text{i.e.} \quad M = M_\alpha \alpha + M_\beta \beta.\]

We make use of the following fact of tensor algebra: for any vector \(M\) there exists a vector \(\eta\) such that the equality holds

\[D_{\alpha\beta} = D_\alpha \alpha \otimes \alpha + D_\beta \beta \otimes \beta = D_M M \otimes M + D_\eta \eta \otimes \eta.\]

Indeed, let the operator \(D_{\alpha\beta}\) act on the vectors \(D^{-1}_{\alpha\beta} M\) and \(D^{-1}_{\alpha\beta} \eta\). We arrive at the following relations for the determination of the vector \(\eta\) and the coefficients \(D_M\) and \(D_\eta\):

\[(M, D^{-1}_{\alpha\beta} \eta) = 0, \quad D_M = (M, D^{-1}_{\alpha\beta} M)^{-1}, \quad D_\eta = (\eta, D^{-1}_{\alpha\beta} \eta)^{-1}.\]

For definiteness we set

\[\eta = D_\alpha M_\beta \alpha - D_\beta M_\alpha \beta.\]

Then, using (33), one can show that at the level of the integral \(E = (M, \omega)\) the map

\[\tilde{\gamma} = k^{-1} \eta, \quad \tilde{M} = (1 + \varepsilon D_M)M = J \omega - D_\eta k^{-2} \tilde{\gamma} \otimes \tilde{\gamma}, \quad k^2 = D_\alpha^2 M_\beta^2 + D_\beta^2 M_\alpha^2, \quad D_M = D_\eta D_\alpha D_\beta, \quad D_\eta = (D_\alpha M_\beta^2 + D_\beta M_\alpha^2)^{-1}\]

reduces the system to the Chaplygin ball problem at the joint level of the integrals

\[(M, \gamma) = k^{-1}(1 + \varepsilon D_M)(D_\alpha - D_\beta)M_\alpha M_\beta, \quad \tilde{C} = (1 + \varepsilon D_M)^2 \tilde{C}, \quad \tilde{E} = (1 + \varepsilon D_M)E.\]

2.5. Linear Trajectory Isomorphism

It follows from Eqs. (32) that the vectors \(M, \alpha\) and \(\beta\) are constant in a fixed frame of reference, therefore their linear combinations with constant (on trajectories) coefficients also remain fixed and are governed by the same equations:

\[\tilde{M} = m_1 M + m_2 \alpha + m_3 \beta, \quad \tilde{\alpha} = a_1 M + a_2 \alpha + a_3 \beta, \quad \tilde{\beta} = b_1 M + b_2 \alpha + b_3 \beta, \quad \tilde{\omega} = \tilde{M} \times \omega, \quad \tilde{\alpha} = \alpha \times \omega, \quad \tilde{\beta} = \beta \times \omega.\]

(45)

In addition, as in the case of the Chaplygin transformation, we require that the transformed angular velocity be expressed in terms of the vectors \(\tilde{M}, \tilde{\alpha}\) and \(\tilde{\beta}\) analogously to the initial angular velocity (34) taking into account the change

\[D_\alpha \rightarrow \tilde{D}_\alpha, \quad D_\beta \rightarrow \tilde{D}_\beta.\]

Let us fix the level of the energy integral (37) \(2H = \tilde{E}\) and determine the metric for three-dimensional vectors

\[\langle x, y \rangle = \varepsilon x_1 y_1 + D^{-1}_\alpha x_2 y_2 + D^{-1}_\beta x_3 y_3,\]

then the following theorem holds

**Theorem 2.** If at the fixed level of the energy integral \(2H = \tilde{E}\) the lines of the matrix made up of the transformation coefficients (45)

\[Q_\varepsilon = \begin{vmatrix} m_1 & m_2 & m_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix},\]

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are mutually orthogonal relative to the metric (46) and the constants $\tilde{D}_\alpha$ and $\tilde{D}_\beta$ are determined as

\begin{align}
\langle m, a \rangle &= \langle m, b \rangle = \langle a, b \rangle = 0, \\
\tilde{D}_\alpha^{-1} &= \langle a, a \rangle, \quad \tilde{D}_\beta^{-1} = \langle b, b \rangle,
\end{align}

(47)

then the vectors $\omega$ and $\tilde{\omega}$ are proportional:

$$
\omega = \lambda \tilde{\omega},
$$

where $\lambda$ is the function of the vectors $M$, $\alpha$, $\beta$, and $\tilde{\omega}$ is expressed in terms of $\tilde{M}$, $\tilde{\alpha}$, $\tilde{\beta}$ using the formula similar to (34).

Proof. We do not present here full calculations (they can be performed by means of a system of analytical calculations, for instance, Maple, Mathematica etc.), we point out only the main stages of the proof.

We denote the elements of the matrix in formula (34) as follows:

$$
X_{\alpha\alpha} = D_\alpha^{-1} - (\tilde{\alpha}, A\alpha), \quad X_{\beta\beta} = D_\beta^{-1} - (\beta, A\beta), \quad X_{\alpha\beta} = -(\alpha, A\beta).
$$

(48)

At the level of the energy integral $2H = \mathcal{E}$ the following relation holds

$$(A\tilde{M}, M) = \mathcal{E} - (Y, X^{-1}Y),$$

where $X = ||X_{\mu\nu}||$ is a $2 \times 2$ function of the elements (48) and $Y = (Y_\alpha, Y_\beta)$ is a two-dimensional vector.

From the condition of conservation of the dependence $\tilde{\omega}(\tilde{M}, \tilde{\alpha}, \tilde{\beta})$

$$
\tilde{\omega} = A(\tilde{M} + \tilde{Z}_\alpha \tilde{\alpha} + \tilde{Z}_\beta \tilde{\beta}), \quad \tilde{Z} = \tilde{X}^{-1}\tilde{Y},
$$

using the transformations (45) and the relations (47) we obtain

$$
\begin{align}
\tilde{X}_{\alpha\alpha} &= \tilde{D}_\alpha^{-1} - (\tilde{\alpha}, A\tilde{\alpha}) = a_1^2(Y, X^{-1}Y) + a_2^2X_{\alpha\alpha} + a_3^2X_{\beta\beta} + 2a_2a_3X_{\alpha\beta} - 2a_1a_2Y_\alpha - 2a_1a_3Y_\beta, \\
\tilde{X}_{\beta\beta} &= \tilde{D}_\beta^{-1} - (\tilde{\beta}, A\tilde{\beta}) = b_1^2(Y, X^{-1}Y) + b_2^2X_{\alpha\alpha} + b_3^2X_{\beta\beta} + 2b_2b_3X_{\alpha\beta} - 2b_1b_2Y_\alpha - 2b_1b_3Y_\beta, \\
\tilde{X}_{\alpha\beta} &= -(\tilde{\alpha}, A\tilde{\beta}) = a_1b_1(Y, X^{-1}Y) + a_2b_2X_{\alpha\alpha} + a_3b_3X_{\beta\beta} + (a_2b_3 + a_3b_2)X_{\alpha\beta} \\
&\quad - (a_1b_2 + a_2b_1)Y_\alpha - (a_1b_3 + a_3b_1)Y_\beta, \\
\tilde{Y}_\alpha &= -A(\tilde{M}(\tilde{\alpha}), A\tilde{\alpha}) = m_1a_1(Y, X^{-1}Y) + m_2a_2X_{\alpha\alpha} + m_3a_3X_{\beta\beta} + (m_2a_3 + m_3a_2)X_{\alpha\beta} \\
&\quad - (m_1a_2 + m_2a_1)Y_\alpha - (m_1a_3 + m_3a_1)Y_\beta, \\
\tilde{Y}_\beta &= A(\tilde{M}(\tilde{\beta}), A\tilde{\beta}) = m_1b_1(Y, X^{-1}Y) + m_2b_2X_{\alpha\alpha} + m_3b_3X_{\beta\beta} + (m_2b_3 + m_3b_2)X_{\alpha\beta} \\
&\quad - (m_1b_2 + m_2b_1)Y_\alpha - (m_1b_3 + m_3b_1)Y_\beta.
\end{align}
$$

We note that, as in the case of the Chaplygin ball, it follows from conditions (47) that $\tilde{X}$ and $\tilde{Y}$ are homogeneous functions of $X$ and $Y$.

Since the proportionality conditions for the vectors $\omega$ and $\tilde{\omega}$ are equivalent to the proportionality condition for the vectors $J\omega$ and $J\tilde{\omega}$, we calculate these vectors in the fixed basis $\alpha, \beta, \gamma = \alpha \times \beta$:

$$
\begin{align}
J\omega &= M + Z_\alpha \alpha + Z_\beta \beta = (M_\alpha + Z_\alpha)\alpha + (M_\beta + Z_\beta)\beta + M_\gamma \gamma, \\
J\tilde{\omega} &= \tilde{M} + \tilde{Z}_\alpha \tilde{\alpha} + \tilde{Z}_\beta \tilde{\beta} = \lambda^{-1}[(M_\alpha + Z_\alpha)\alpha + (M_\beta + Z_\beta)\beta + M_\gamma \gamma],
\end{align}
$$

\begin{align}
Z_\alpha &= \frac{X_{\beta\beta}Y_\alpha - X_{\alpha\beta}Y_\beta}{X_{\alpha\alpha}X_{\beta\beta} - X_{\alpha\beta}^2} \quad Z_\beta = \frac{X_{\alpha\beta}Y_\alpha - X_{\alpha\gamma}Y_\gamma}{X_{\alpha\alpha}X_{\beta\beta} - X_{\alpha\beta}^2},
\end{align}

where $M_\alpha = (M, \alpha)$, $M_\beta = (M, \beta)$, $M_\gamma = (M, \gamma)$,

$$
\lambda = \frac{a_2b_3 - a_3b_2}{\det Q_\xi} + \frac{a_3b_1 - a_1b_3}{\det Q_\xi} Z_\alpha + \frac{a_1b_2 - a_2b_1}{\det Q_\xi} Z_\beta.
$$
furthermore, the quantities $\tilde{Z}_a$ and $\tilde{Z}_\beta$ are related to $Z_a$ and $Z_\beta$ by the following linear transformation:

$$
Z_a = \lambda (m_2 + a_2 \tilde{Z}_a + b_2 \tilde{Z}_\beta),
$$
$$
Z_\beta = \lambda (m_3 + a_3 \tilde{Z}_a + b_3 \tilde{Z}_\beta).
$$

Thus, as above, the vector field of the transformed system differs from the vector field of the original system (32) by the scalar factor

$$
\hat{M} = \lambda \tilde{M} \times (A \tilde{M} + \tilde{Z}_a \tilde{\alpha} + \tilde{Z}_\beta \tilde{\beta}),
$$

$$
\hat{\alpha} = \lambda \tilde{\alpha} (A \tilde{M} + \tilde{Z}_a \tilde{\alpha} + \tilde{Z}_\beta \tilde{\beta}),
$$

$$
\hat{\beta} = \lambda \tilde{\beta} (A \tilde{M} + \tilde{Z}_a \tilde{\alpha} + \tilde{Z}_\beta \tilde{\beta}),
$$

and the new level of the energy integral, as can be verified using (47), is given by the equation

$$
\tilde{E} = (\tilde{M}, \tilde{\omega}) = \mathcal{E} m^2_1 + \mathcal{D}_a^{-1} m_2^2 + \mathcal{D}_\beta^{-1} m_3^2 = \langle m, m \rangle.
$$

2.6. Reduction to Zero “Constant of Areas”

We show that using the transformation $Q_\mathcal{E}$ described in the previous section one can choose the new vectors $\tilde{M}, \tilde{\alpha}$ and $\tilde{\beta}$ to be mutually orthogonal:

$$
(\tilde{M}, \tilde{\alpha}) = (\tilde{M}, \tilde{\beta}) = (\tilde{\alpha}, \tilde{\beta}) = 0. \tag{49}
$$

(At the first stage we need not require them to be normalized.)

Let $\Gamma$ be the Gram matrix of the initial vectors $M, \alpha, \beta$ and $g$ be the metric matrix $\langle \cdot, \cdot \rangle$:

$$
\Gamma = \begin{bmatrix}
C & M_\alpha & M_\beta \\
M_\alpha & 1 & 0 \\
M_\beta & 0 & 1
\end{bmatrix},
\quad
\Gamma^{-1} = \begin{bmatrix}
\mathcal{E} & 0 & 0 \\
0 & \mathcal{D}_a^{-1} & 0 \\
0 & 0 & \mathcal{D}_\beta^{-1}
\end{bmatrix},
\tag{50}
$$

where $M_\alpha = (M, \alpha)$, $M_\beta = (M, \beta)$, $C = (M, M)$.

Proposition 2. Let $B = (\Gamma g^{-1})^T$ and $m, a, b$ be its eigenvectors corresponding to different eigenvalues:

$$
Bm = \lambda_m m, \quad Ba = \lambda_a a, \quad Bb = \lambda_b b. \tag{51}
$$

Then the matrix $Q_\mathcal{E}$ of transformation (45) whose lines consist of the elements of the vectors $m, a$ and $b$ simultaneously satisfy conditions (47) and (49).

Proof. The requirement (49), along with conditions (47), leads to the following matrix equalities, which must be satisfied by the matrix $Q_\mathcal{E}$:

$$
Q_\mathcal{E} \Gamma Q_\mathcal{E}^T = \tilde{\Gamma},
$$

$$
Q_\mathcal{E} g Q_\mathcal{E}^T = \tilde{g}.
$$

where $\tilde{\Gamma} = \text{diag}(\tilde{M}^2, \tilde{\alpha}^2, \tilde{\beta}^2)$, $\tilde{g} = \text{diag}(\tilde{E}, \tilde{D}_a^{-1}, \tilde{D}_\beta^{-1})$.

From the second relation (52) we find

$$
(Q_\mathcal{E}^T)^{-1} = \tilde{g}^{-1} Q_\mathcal{E} g;
$$

substituting into the first of the relations (52), we obtain the matrix equation

$$
Q_\mathcal{E} \Gamma g^{-1} = \tilde{\Gamma} \tilde{g}^{-1} Q_\mathcal{E}. \tag{53}
$$

Denoting the unknown $\lambda_m = \tilde{E}^{-1} \tilde{M}^2$, $\lambda_a = \tilde{D}_a \tilde{\alpha}^2$ and $\lambda_b = \tilde{D}_\beta \tilde{\beta}^2$, we immediately verify that (53) is equivalent to Eqs. (51).
Thus, we have proved that subject to conditions (47) and (49) the matrix \( Q_\mathcal{E} \) consists of the vectors \( \mathbf{m}, \mathbf{a} \) and \( \mathbf{b} \) given by (51). In order to prove the converse, we write the transformation \( Q_\mathcal{E} \) in explicit form using (50) and (51),

\[
Q_\mathcal{E} = \begin{pmatrix}
m_1 \frac{M_\alpha D_\alpha}{\lambda_1 - D_\alpha} m_1 & M_\beta D_\beta \frac{M_\alpha D_\alpha}{\lambda_1 - D_\alpha} m_1 \\
a_1 \frac{M_\alpha D_\alpha}{\lambda_2 - D_\alpha} a_1 & M_\beta D_\beta \frac{M_\alpha D_\alpha}{\lambda_2 - D_\beta} a_1 \\
1 \frac{M_\beta D_\alpha}{\lambda_3 - D_\alpha} b_1 & M_\beta D_\beta \frac{M_\alpha D_\alpha}{\lambda_3 - D_\beta} b_1
\end{pmatrix},
\]

where \( \lambda_m, \lambda_a \) and \( \lambda_b \) are the roots of the cubic equation \( \det(\Gamma - \lambda \mathbf{g}) = 0 \), which is written as

\[
\mathcal{E} \lambda^3 - (\mathcal{E}(D_\alpha + D_\beta) + C) \lambda^2 + (D_\alpha(C - M_\alpha^2) + D_\beta(C - M_\beta^2) + \mathcal{E} D_\alpha D_\beta) \lambda - (C - M_\alpha^2 - M_\beta^2) D_\alpha D_\beta = 0.
\]

It is straightforward to show that the equation \( \det(\Gamma - \lambda \mathbf{g}) = 0 \) in the case of symmetric \( \Gamma \) and \( \mathbf{g} \) has only real roots; furthermore, the sign of the coefficients in front of the degrees \( \lambda \) changes three times, hence by Descartes’ rule, all these roots are positive. Using immediate calculations, substituting the transformation (54) into Eqs. (52), one can verify that the matrices \( \tilde{\Gamma} \) and \( \tilde{\mathbf{g}} \) are diagonal, in which case \( \tilde{\mathcal{E}}, \tilde{\mathcal{D}}_\alpha \) and \( \tilde{\mathcal{D}}_\beta \) are given by (47) and the elements of the matrix have the form

\[
\tilde{\mathbf{M}}^2 = (\mathbf{m}, \tilde{\mathbf{\Gamma}} \mathbf{m}), \quad \tilde{\alpha}^2 = (\mathbf{a}, \tilde{\mathbf{\Gamma}} \mathbf{a}), \quad \tilde{\beta}^2 = (\mathbf{b}, \tilde{\mathbf{\Gamma}} \mathbf{b}),
\]

where the vectors \( \mathbf{m}, \mathbf{a} \) and \( \mathbf{b} \) are given by the matrix (54).

It follows from (54) that three more constants \( m_1, a_1 \) and \( b_1 \) remain undetermined, which can be used to normalize the vectors \( \tilde{\mathbf{M}}, \tilde{\alpha} \) and \( \tilde{\beta} \).

Thus, finally we obtain that after transformation of (54) at the level of energy \( (\mathbf{M}, \mathbf{\omega}) = \mathcal{E} \) the system reduces to the case described in Section 2.4 and, consequently, by Proposition 1, can be explicitly integrated by quadratures.

2.7. Bifurcation Analysis and Stability of Periodic Solutions

a. The Case \( M \parallel \beta \). In this case the constants of the first integrals

\[
\alpha^2 = \beta^2 = 1, \quad (\alpha, \beta) = (\mathbf{M}, \alpha) = (\mathbf{M}, \alpha \times \beta) = 0, \\
(\mathbf{M}, \beta) = M_\beta, \quad (\mathbf{M}, \mathbf{M}) = M_\beta^2, \quad (\mathbf{M}, \mathbf{\omega}) = \mathcal{E}.
\]

By Proposition 1, in this case the critical periodic trajectories are determined by critical solutions of the Chaplygin system at a zero constant of areas (see Section 1.5). In the notation of Proposition 1 the bifurcation curves are given by

\[
\tilde{\mathcal{E}} = A_i \tilde{\mathcal{C}}, \quad i = 1, 2, 3, \quad \tilde{\mathcal{C}} = (\tilde{\mathbf{M}}, \tilde{\mathbf{\omega}}), \quad \tilde{\mathcal{C}} = (\tilde{\mathbf{M}}, \tilde{\mathbf{M}}).
\]

After simplifications using relations (33) and (44) we express them in terms of the initial integrals

\[
\mathcal{E} = \frac{A_i}{1 - D_\beta A_i} M_\beta^2 = \frac{M_\beta^2}{J_i^{(0)} - D_\beta^{(0)}},
\]

where \( J_i^{(0)} \) are the elements of the diagonal matrix \( \mathbf{J}^{(0)} = \mathbf{I} + \left( \sum_k \frac{w_k}{r_k} \right) \mathbf{E} \) with \( M_\beta \in (-\infty, +\infty) \).

For these solutions the angular velocity of the central ball \( \mathbf{\omega} \) is directed along one of its principal axes \( e_i \) and is constant in the fixed axes, so that

\[
\mathbf{M} = M_\beta e_i, \quad \mathbf{\omega} = M_\beta A_i e_i, \quad \beta = e_i, \quad \alpha = \cos \varphi e_j + \sin \varphi e_k, \quad \varphi = A_i M_\beta t,
\]

where \( i, j, k \) is a cyclic permutation of the numbers 1, 2, 3.
b. The general case. The projections of the angular momentum onto the fixed axes are arbitrary in this case:

\[(M, \alpha) = M_\alpha,\quad (M, \beta) = M_\beta,\quad (M, \gamma) = M_\gamma,\]
\[(M, M) = C = M^2_\alpha + M^2_\beta + M^2_\gamma.\]

We make use of the existence of linear isomorphism described in Sections 2.5 and 2.6. By Theorem 2 and Proposition 2 the angular velocity \(\omega\) for critical solutions in the general case is proportional to the angular velocity of critical solutions at \(M\|\beta\), consequently, the following relations hold

\[\omega = \Lambda A_i e_i,\quad M + Z_\alpha \alpha + Z_\beta \beta = \Lambda e_i,\quad i = 1, 2, 3.\]

The scalar multiplication of the first of these equations by the vectors \(\alpha, \beta, M, e_i\), using (31) and (34) gives the equations

\[(\omega, \alpha) = D^{-1}_\alpha Z_\alpha = \Lambda A_i \alpha_i,\quad (\omega, \beta) = D^{-1}_\beta Z_\beta = \Lambda A_i \beta_i,\]
\[M_\alpha + Z_\alpha = \Lambda \alpha_i,\quad M_\beta + Z_\beta = \Lambda \beta_i,\]
\[C + Z_\alpha M_\alpha + Z_\beta M_\beta = \Lambda M_i,\quad M_i + Z_\alpha \alpha_i + Z_\beta \beta_i = \Lambda.\]

Solving them, we find

\[\Lambda^2 = C + \frac{(2D^{-1}_\alpha - A_i)A_i}{(D^{-1}_\alpha - A_i)^2} M^2_\alpha + \frac{(2D^{-1}_\beta - A_i)A_i}{(D^{-1}_\beta - A_i)^2} M^2_\beta,\]
\[\Lambda \alpha_i = \frac{D^{-1}_\alpha}{D^{-1}_\alpha - A_i} M_\alpha,\quad \Lambda \beta_i = \frac{D^{-1}_\beta}{D^{-1}_\beta - A_i} M_\beta,\]
\[Z_\alpha = \frac{A_i}{D^{-1}_\alpha - A_i} M_\alpha,\quad Z_\beta = \frac{A_i}{D^{-1}_\beta - A_i} M_\beta.\]

Substituting into the relations \(E = (M, \omega) = \Lambda A_i M_i\), we find the equations of bifurcation surfaces in the form

\[E = A_i C + \frac{A^2_i}{D^{-1}_\alpha - A_i} M^2_\alpha + \frac{A^2_i}{D^{-1}_\beta - A_i} M^2_\beta = \frac{M^2_\alpha}{J_\alpha^{(0)} - D_\alpha^{(0)}} + \frac{M^2_\beta}{J_\beta^{(0)} - D_\beta^{(0)}} + \frac{M^2_\gamma}{J_\gamma^{(0)} - D_\gamma^{(0)}}.\]

By analogy with the Chaplygin problem we determine the variables \(E = \frac{g}{\sqrt{C}}, g_\alpha = \frac{M_\alpha}{\sqrt{C}}, g_\beta = \frac{M_\beta}{\sqrt{C}}, g_\gamma = \frac{M_\gamma}{\sqrt{C}}\), then the bifurcation surfaces are determined as the quadratic functions

\[
\frac{g^2_\alpha}{J_\alpha^{(0)} - D_\alpha^{(0)}} + \frac{g^2_\beta}{J_\beta^{(0)} - D_\beta^{(0)}} + \frac{g^2_\gamma}{J_\gamma^{(0)} - D_\gamma^{(0)}} = E
\]

on the unit sphere

\[g^2_\alpha + g^2_\beta + g^2_\gamma = 1.\]

A typical bifurcation diagram projected onto the plane \(g_\alpha, g_\beta\) is shown in Fig. 5.

3. THE MOTION OF THE CHAPLYGIN BALL ALONG A STRAIGHT LINE (VESELOVS’ SYSTEM)

In conclusion we consider the Chaplygin ball problem with an additional constraint forcing a ball to roll along a straight line [19]. Like the problems considered above, the equations of motion reduce to a system of nine differential equations

\[
\dot{K} = K \times \omega,\quad \dot{\alpha} = \alpha \times \omega,\quad \dot{\beta} = \beta \times \omega,\quad K = (I + D\beta \otimes \beta - \alpha \otimes I\alpha)\omega.
\]

In addition to the energy integral and obvious geometric integrals, as shown in [19], these equations admit an additional (quadratic in momenta) integral of motion and an integrating multiplier. Thus, Eqs. (57) are integrable by the Euler–Jacobi theorem. Later, in our uncompleted
book [4] it was shown that the integral quadratic in momenta can be represented as a combination of linear integrals. Thus, a complete set of integrals for the system (57) can be written as

\[
\begin{align*}
(\alpha, \alpha) &= (\beta, \beta) = 1, \\
(\alpha, \beta) &= 0, \\
(K, \alpha) &= 0, \\
(K, \beta) &= \text{const}, \\
(K, \gamma) &= (K, \alpha \times \beta) = \text{const},
\end{align*}
\]

(58)

In the case where one of the constants of areas is zero ((\(K, \beta\)) = 0 or (\(K, \gamma\)) = 0), Eqs. (57) can be reduced to the equations governing the rolling of a ball on a plane without slipping and spinning (rubber ball).

1. Let (\(K, \beta\)) = 0 (\(K\|\gamma\)). Then, after introducing the vector

\[
\hat{K} = K + D \frac{\mathcal{E}}{(K, \gamma)^2} K = (I - \alpha \otimes I \alpha) \omega,
\]

where \(J = I + D \mathcal{E}\), we obtain a system of equations closed relative to \(\hat{K}\), \(\alpha\) that governs the rolling of a rubber ball with tensor of inertia \(J\).

2. In a similar fashion, in the case (\(K, \gamma\)) = 0 (\(K\|\beta\)), introducing the vector

\[
\hat{K} = K - D \frac{\mathcal{E}}{(K, \beta)^2} K = (I - \alpha \otimes I \alpha) \omega,
\]

we obtain a system closed relative to \(\hat{K}\), \(\alpha\) that governs the rolling of a rubber ball with tensor of inertia \(I\).

As is well known, the rubber ball rolling problem is equivalent to the Veselova problem (the motion of a rigid body about a fixed point with the non-holonomic constraint (\(\omega, \gamma\)) = 0, i.e. a zero projection of the angular velocity onto one of the fixed axes) [20] and can be explicitly integrated in spherico-conical coordinates. Thus, we have proved that in the case where one of the constants of areas is zero the problem under consideration can also be explicitly integrated in spherico-conical coordinates.

In the general case where none of the constant of areas is zero, all attempts at solving explicitly Eqs. (57) (at least to our knowledge) have not been successful. In this connection it would be interesting to deal with the problem of generalizing the transformation discussed in this paper and to use it for explicit integration of Veselovs’ problem for arbitrary constants of areas.

ACKNOWLEDGMENTS

The authors are grateful to A. V. Bolsinov, Yu. N. Fedorov, A. V. Tsiganov and A. Yu. Moskvin for useful discussions.

This research was supported by the Grant of the Government of the Russian Federation for state support of scientific research conducted under supervision of leading scientists in Russian educational institutions of higher professional education (contract no. 11.G34.31.0039) and the
Federal target programme “Scientific and Scientific-Pedagogical Personnel of Innovative Russia” measure 1.5 “Topology and Mechanics” (project code 14.740.11.0876). The work was supported by the Grant of the President of the Russian Federation for the Leading Scientific Schools of the Russian Federation (NSh-2519.2012.1).

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