The paper studies the system of a rigid body interacting dynamically with point vortices in a perfect fluid. For arbitrary value of vortex strengths and circulation around the cylinder the system is shown to be Hamiltonian (the corresponding Poisson bracket structure is rather complicated). We also reduced the number of degrees of freedom of the system by two using the reduction by symmetry technique and performed a thorough qualitative analysis of the integrable system of a cylinder interacting with one vortex.

In this paper we consider the system of a rigid cylinder interacting with point vortices. We start by indicating some known results on the subject from the classical hydrodynamics. For the most part, these results are presented in [18, 6, 9].

As far as we can see, Kirchhoff [5] was the first who studied the dynamics of point vortices on a systematic basis. In particular, he obtained the equations of motion in Hamiltonian form and indicated integrals of motion. It was enough to show that the equations of motion governing the system of two or three vortices are integrable (the problem of three vortices was first analytically solved by Gröbli).

The classics also considered the system of point vortices moving externally to rigid, stationary boundaries (the impermeable condition on the boundaries was assumed). Greenhill [12] studied in detail the motion of two vortices in a circular region. Havelock [13] investigated stability of n-gon stationary configurations in the region exterior to a circle. We should also mention Föppl’s analysis of interaction of a rigid body with an ambient flow at low Reynolds numbers (more exactly, he investigated stability of the system of two vortices interacting with a circular cylinder embedded in a uniform flow).

At the same time, the study of the dynamics of interacting vortices-body systems has been also a subject of interest in the classical hydrodynamics. It is known from the phenomenological theory developed by Prandtl [7] and Jukowski that when a body moves in a fluid, the thin boundary layer peels off the body thereby generating bound vortices. In their turn, these vortices exert a lifting force on the body which can be observed in aero- and hydrodynamical experiments.

The problem of interaction of a rigid body and point vortices in a perfect fluid (the impermeable condition on the body’s surface is assumed) can be studied within the framework of the Hamiltonian
mechanics. Various forms of the equations of motion for a rigid circular cylinder interacting with $n$ point vortices have been recently (and practically simultaneously) obtained in [14, 15, 16]. The integrability of the equations in the case of one vortex ($n = 1$) was established in [10].

In this paper we will study in greater detail this case of integrability. We will also consider the simplest chaotic system of a cylinder and two vortices to which a reduction procedure will be applied resulting in reduction of degrees of freedom by one.

Hereinafter, as in [10], the term rigid body will refer to a two-dimensional circular region. It should be noted that even in the case of an elliptic region the equations of motion become much more complicated and cannot be written in such a compact form.

1. Hamiltonian form of the equations of motion

The equations of motion for a cylinder and vortices with respect to a fixed coordinate frame $Oxy$ can be written as [14]

$$\dot{r}_i = -v + \text{grad} \tilde{\varphi}_i_{|_{r=r_i}}, \quad \dot{r}_c = v$$

$$a\dot{v}_1 = \lambda v_2 - \sum_{i=1}^{n} \lambda_i (\tilde{y}_i - y_i), \quad a\dot{v}_2 = -\lambda v_1 + \sum_{i=1}^{n} \lambda_i (\tilde{x}_i - x_i),$$

where $r_c$ is the radius-vector from $O$ to the center of mass of the cylinder, $v$ is the velocity of the cylinder, $r_i$ is the vector from the center of the cylinder to the $i$-th vortex and $\dot{r}_c$ is the vector from the center of the cylinder to the $i$-th inverse point (Fig. 1). Here $R$ denotes the radius of the cylinder, the constant coefficient $a$ involves the added mass of the cylinder; and the constants $\lambda$ and $\lambda_i$ are connected with the circulation around the cylinder and the vortex strengths by the formulae $\lambda = \frac{\Gamma}{2\pi}, \lambda_i = \frac{\Gamma_i}{2\pi}$. The density of the fluid is $2\pi$.

![Fig. 1](image)

The function $\tilde{\varphi}_i(r)$ represents that portion of the velocity potential $\varphi(r)$ which does not have a singularity at the point $r = r_i$. The velocity potential in the region exterior to the cylinder reads

$$\varphi(r) = -\frac{R^2}{r^2} (r, v) - \lambda \arctg \frac{y}{x} + \sum_{i=1}^{n} \lambda_i \left( \arctg \left( \frac{y - y_i}{x - x_i} \right) - \arctg \left( \frac{y - \tilde{y}_i}{x - \tilde{x}_i} \right) \right).$$

Equations (1.1) were derived using a balance of linear momentum for the fluid within a circular boundary that encloses the body and the vortices. The fluid is assumed to be at rest at infinity [14, 16].

Thus the analysis of the system of a cylinder and vortices in a perfect fluid can be reduced to the analysis of a finite set of ordinary differential equations. It is easy to check that equations (1.1) preserve invariant measure, i.e., the divergence of (1.1) is zero.
As for the system of \( n \) point vortices [5], equations (1.1) can be shown to be Hamiltonian.

**Proposition 1.** Equations (1.1) can be represented in the form

\[
\dot{\zeta}_i = \{\zeta_i, H\} = \sum_k \{\zeta_i, \zeta_k\} \frac{\partial H}{\partial \zeta_k}
\]  

(1.3)

where \( \zeta_i \) are the components of the phase vector \( \zeta = (x_c, y_c, v_1, v_2, x_1, y_1, \ldots, x_n, y_n) \) and \( H \) is the Hamiltonian. For the components \( J_{ij} (\zeta) = \{\zeta_i, \zeta_j\} \) of the structural tensor of the Poisson bracket structure the Jacobi identity holds:

\[
\sum_l \left( J_{jl} \frac{\partial J_{lk}}{\partial \zeta_l} + J_{kl} \frac{\partial J_{lj}}{\partial \zeta_l} + J_{lj} \frac{\partial J_{kl}}{\partial \zeta_l} \right) = 0 \quad \forall i, j, k
\]

**Proof.**

Equations (1.1) has an integral of motion:

\[
H = \frac{1}{2} a v^2 + \frac{1}{2} \sum_i \left( \lambda_i^2 \ln(r_i^2 - R^2) - \lambda_i \lambda_n r_i^2 \right) + \frac{1}{2} \sum_{i<j} \lambda_i \lambda_j \ln \frac{R^2 - 2R^2(r_i, r_j) + r_i^2 r_j^2}{|r_i - r_j|^2}.
\]

(1.4)

This integral resembles the Hamiltonian of the system of \( n \) point vortices [5].

Let \( H \) be our Hamiltonian. Now we have to find a skew-symmetric tensor \( J_{ij} \) such that the equations of motion (1.3) coincide with (1.1). The non-zero components of this tensor are

\[
\{v_1, x_i\} = \frac{1}{a} \frac{r_i^4 - R^2(x_i^2 - y_i^2)}{r_i^4}, \quad \{v_1, y_i\} = \frac{1}{a} \frac{2R^2 x_i y_i}{r_i^4},
\]

\[
\{v_2, x_i\} = -\frac{1}{a} \frac{2R^2 x_i y_i}{r_i^4}, \quad \{v_2, y_i\} = \frac{1}{a} \frac{r_i^4 + R^2(x_i^2 - y_i^2)}{r_i^4},
\]

(1.5)

\[
\{v_1, v_2\} = \frac{\lambda}{a^2} - \sum_i \frac{\lambda_i r_i^4 - R^4}{r_i^4}, \quad \{x_i, v_1\} = -\frac{1}{\lambda_i},
\]

\[
\{x_c, v_1\} = \{y_c, v_2\} = a^{-1}.
\]

It is easy to verify that for the Poisson bracket (1.5) the Jacobi identity holds.

The Poisson bracket structure (1.5) is non-degenerate, therefore, by the Darboux theorem, it can be reduced to a canonical form \( \{q_i, p_j\} = \delta_{ij} \). However, in our further analysis canonical coordinates will not be used.

The Lie-Poisson bracket structure for (1.1), (1.2) under the condition \( \lambda = -\sum \lambda_i \) was studied in [16]. In this work stability of an equilibrium configuration of the system of two vortices behind a steadily moving cylinder is discussed. The Poisson bracket (1.5) was first obtained in [10].

The fact that the general equations are Hamiltonian is not a priori obvious and does not seem to follow from the Lagrangian formalism. The Hamiltonian form of the equations allows us to apply the highly developed perturbation theory techniques (e.g. the KAM theory) and other specific methods of qualitative analysis. The Liouville theorem on integrability and its geometrical extension suggested by Arnold [1] can be applied to the analysis of these equations.

2. Problem of advection

The problem of finding pathlines of the fluid for a given motion of the cylinder \( r_c = r_c(t) \) and the vortices \( r_i = r_i(t), i = 1, \ldots, n \) is known as the problem of advection. As mentioned above, the
ambient flow is potential with potential (1.2), therefore the equation of motion for a passive particle with respect to the cylinder-fixed frame of reference looks like

$$\dot{r} = \text{grad} \varphi(r)|_{r_i=r_i(t)}, \quad r = (x, y).$$  \hfill (2.1)

Obviously, equations (2.1) are Hamiltonian with respect to the standard Poisson bracket structure \(\{x; y\} = 1\); the Hamiltonian is time-dependent and coincides with the stream function for the potential (1.2), that is,

$$H_a(r, t) = \left(\frac{R^2}{r^2} - 1\right) (v_1(t)y - v_2(t)x) + \frac{1}{2} \lambda \ln r^2 +$$

$$\frac{1}{2} \sum_{i=1}^{N} \lambda_i (\ln |r - r_i(t)|^2 - \ln |r - \tilde{r}_i(t)|^2).$$  \hfill (2.2)

3. Symmetry and integrals of motion

The equations of motion (1.1) are invariant under the action of the Euclidean group \(E(2)\), therefore, by Noether’s theorem for Hamiltonian systems, there exist three integrals of motion.

The integrals

$$Q = av_2 + \lambda x_c - \sum \lambda_i (\tilde{x}_i - x_i), \quad P = av_1 - \lambda y_c + \sum \lambda_i (\tilde{y}_i - y_i)$$  \hfill (3.1)

correspond to translations along the coordinate axes. These integrals are a generalization to the classical linear momentum. On the other hand, the vector \((Q, P)\) can be considered as a counterpart of the vector of the center of vorticity of the system of \(n\) vortices and cylinder. For the system of \(n\) vortices this notion was introduced in [5, 6]. For \(\lambda \neq 0\), the origin of coordinates can be so chosen as to make \(Q = P = 0\), meaning that the center of vorticity can be always shifted to the origin of coordinates.

The third integral, corresponding to the invariance under rotations about an axis perpendicular to the plane of motion, is

$$I = a(v_1 y_c - v_2 x_c) - \frac{1}{2} \lambda r_c^2 - \frac{1}{2} \sum_{i=1}^{n} \lambda_i r_i^2 + \frac{1}{2} \sum_{i=1}^{n} \left(\frac{R^2}{r_i^2} - 1\right) (r_i, r_c).$$  \hfill (3.2)

In the paper [10], an additional integral for equations (1.1) but with the dynamics for the center of the cylinder excluded was indicated. This integral looks like

$$F = a^2 v^2 + \sum_{i=1}^{n} \lambda_i \left(2a \left(1-\frac{R^2}{r_i^2}\right) (x_i v_2 - y_i v_1) + (\lambda_i - \lambda) r_i^2 + \lambda_i \frac{R^4}{r_i^4}\right) +$$

$$+ 2 \sum_{i<j} \lambda_i \lambda_j (r_i, r_j) \left(1-\frac{R^2}{r_i^2}\right) \left(1-\frac{R^2}{r_j^2}\right).$$  \hfill (3.3)

The integrals \(F, I, P\) and \(Q\) satisfy the equation

$$F = 2\lambda I + P^2 + Q^2 + 2R^2 \sum_{i=1}^{n} \lambda_i^2.$$

Comment. The Hamiltonian vector field that corresponds to the integral (3.2) looks like

$$X_I = \{\zeta, I\} = (y_c, -x_c, v_2, -v_1, y_1, -x_1, \ldots, y_n, -x_n)$$  \hfill (3.4)
The Poisson bracket of the integrals \( Q, P, \) and \( I \) differs from the Lie-Poisson bracket for the \( e(2) \) algebra by a constant (co-cycle [1]), that is,

\[
\{ Q, P \} = \lambda, \quad \{ I, Q \} = P, \quad \{ I, P \} = -Q.
\] (3.5)

Thus, for \( \lambda \neq 0 \) the number of degrees of freedom of the system governed by (1.1) can be reduced by two and even by three if \( \lambda = 0 \) and \( P = Q = 0 \).

**Corollary 1.** The dynamics of a cylinder and one vortex is integrable in the Liouville sense.

**Corollary 2.** For \( \lambda = 0 \) and \( P = Q = 0 \), the dynamics of a cylinder and two vortices is integrable in the Liouville sense.

### 4. Complex form of equations of motion and the Dirac bracket

As mentioned above, in [10] reduced equations were used, i.e., the equations we dealt with were exactly equations (1.1) but without the equation \( \dot{r}_c = \mathbf{v} \). The elimination of this equation from the general system (1.1) now can be interpreted as the reduction by symmetry due to the integrals \( Q \) and \( P \) (3.1). For \( \lambda \neq 0 \) the reduction can be carried out by restricting the dynamics to a joint level surface of the integrals (3.1). Since \( \lambda \neq 0 \), we assume that the origin of the fixed coordinate frame is at the center of vorticity, i.e., \( P = Q = 0 \). Then, we substitute \( \dot{r}_c \) for \( v_i \) in (3.1). The first order equations in the positional variables result. It is convenient to write these equations in the complex form

\[
a \dot{z}_c = a \mathbf{v} = -i \lambda z_c + i \sum_{j=1}^{n} \lambda_j (\bar{z}_j - z_j)
\]

\[
\dot{z}_k = -\mathbf{v} + \frac{R^2 v_j}{z_k^2} + i \lambda \frac{1}{z_k - \bar{z}_k} + \sum_{j \neq k}^{n} \lambda_j \left( \frac{1}{z_k - z_j} - \frac{1}{z_k - \bar{z}_j} \right),
\] (4.1)

where \( z_c = x_c + iy_c \) and \( z_k = x_k + iy_k \) define the position of the cylinder’s center and the vortices, and \( v = v_1 + iv_2 \) is the velocity of the cylinder’s center.

Obviously, equations (4.1) are Hamiltonian. The Hamiltonian can be obtained by replacing the cylinder’s velocity in (1.4) with the expression in the right-hand side of the first equation of (4.1)

\[
H = \frac{1}{2a} \left( \lambda x - \sum_{i=1}^{n} \lambda_i (\bar{x}_i - x_i) \right)^2 + \frac{1}{2a} \left( \lambda y - \sum_{i=1}^{n} \lambda_i (\bar{y}_i - y_i) \right)^2 + \\
+ \frac{1}{2} \sum_{i} \left( \lambda_i^2 \ln(r_i^2 - R^2) - \lambda_i \ln r_i^2 \right) + \frac{1}{2} \sum_{i<j} \lambda_i \lambda_j \ln \frac{R^4 - 2R^2 (r_i, r_j) + r_i^2 r_j^2}{|r_i - r_j|^2}.
\] (4.2)

The Poisson bracket for (4.1) is

\[
\{ x_c, y_c \} = \frac{1}{\lambda}, \quad \{ x_i, y_i \} = -\frac{1}{\lambda_i}, \quad i = 1, \ldots, n.
\] (4.3)

This bracket can be obtained via the Dirac reduction procedure [11], which consists in restricting the bracket (1.5) to the manifold \( Q = P = 0 \). In other words, for \( \lambda \neq 0 \), the cylinder can be considered as an \((n + 1)\)-th "compound vortex", and equations (4.1) govern the system of \( n + 1 \) vortices.
Comment. The Dirac bracket \( \{ \cdot, \cdot \}_D \) on a manifold \( \mathcal{N}_c \) which is a level surface \( f_i(\zeta) = c_i \), \( c_i = \text{const} \), \( i = 1, \ldots, k \) is given by the formula

\[
\{ g, h \}_D = \{ g, h \} + \sum_{ij} \{ g, f_i \} c_{ij} \{ h, f_j \},
\]

where \( \| c_{ij} \| = \| \{ f_i, f_j \} \|^{-1} \) and \( \{ \cdot, \cdot \} \) is the original Poisson bracket. In our case we have

\[
\{ g, h \}_D = \{ g, h \} - \frac{1}{\lambda} \left( \{ g, Q \} \{ h, P \} - \{ g, P \} \{ h, Q \} \right),
\]

where \( \{ \cdot, \cdot \} \) is the Poisson bracket (1.5)

Equations (4.1) are invariant under rotations about the origin (the center of vorticity) and therefore have an additional integral of motion:

\[
I = \lambda r_c^2 - \sum_{i=1}^{n} \lambda_i r_i^2. \tag{4.4}
\]

This integral can be obtained upon substitution of the expression for \( r_c \) from (4.1) for \( \mathbf{v} \) in the integral (3.3).

Now we can formulate some properties of the motion of the system of a cylinder and \( n \) vortices. These properties are analogous to those formulated by Synge [17] for the system of \( n \) vortices.

**Proposition 2.** Suppose that the vortex strengths are of the same sign and \( \lambda \lambda_i < 0 \); then the trajectory of the cylinder’s center and the vortices belong to a bounded region.

**Proposition 3.** Suppose that \( \lambda \neq 0 \) and the vortex motions relative to center of the cylinder are bounded (i.e., \( \forall i \ |r_i(t)| \) is a bounded function), then the absolute motion of the cylinder is also bounded.

Here and in the sequel, a motion with respect to the fixed frame will be referred to as an absolute motion and with respect to the center of the cylinder as a relative motion.

Comment: It should be noted that for \( n = 1 \) and \( \lambda \neq 0 \) a more general statement is valid: the absolute motion of the cylinder is bounded if and only if the relative motion of the vortex is bounded. Simulations have shown that already for \( n = 2 \) this is not true: the trajectories of the cylinder and two vortices for the case \( \lambda_1 = -\lambda_2 \), \( |\lambda| < |\lambda_1| \) are shown in Fig. 2. As might be expected, eventually the vortices start drifting to infinity while the cylinder moves along a curve which gradually takes the shape of a circle.

5. Motion of a cylinder and one vortex

**Integrability of the equations of motion. Reduction to a system with one degree of freedom.** Consider in greater detail the system of a cylinder interacting dynamically with only one vortex, i.e., \( n = 1 \). Let \( r_1 = r = (x, y) \). With the assistance of the first integrals (3.1) and (3.2), the solution to the equations of motion follows in terms of quadratures. Using the integrals, our system can be reduced to a system with one degree of freedom. The algebraic reduction that we will now use is analogous to the Routh reduction.
As the variables of the reduced system, it is reasonable to choose integrals of the field of symmetry
ν (3.4) (see, for example, [2]). We put
\[ p_1 = a(xv_1 + yv_2), \quad p_2 = a(xv_2 - yv_1), \quad \rho = x^2 + y^2. \] (5.1)
The Poisson brackets for these variables are
\[ \{p_1, p_2\} = (\lambda - \lambda_1)\rho + \frac{1}{\lambda_1}(p_1^2 + (p_2 - \lambda_1 R^2))^2, \]
\[ \{p_1, \rho\} = 2\rho + \frac{2}{\lambda_1}(p_2 - \lambda_1 R^2), \quad \{p_2, \rho\} = -2\frac{p_1}{\lambda_1}. \] (5.2)
In terms of (5.1) the integrals (1.4) and (3.3) read
\[ H = \frac{p_1^2 + p_2^2}{2\alpha \rho} + \frac{1}{2\lambda_1^2} \ln (\rho - R^2) - \frac{1}{2}\lambda_1 \lambda \ln \rho, \]
\[ F = \frac{p_1^2}{\rho} + 2\lambda_1 \left( 1 - \frac{R^2}{\rho} \right) p_2 + \lambda_1^2 \left( \rho + \frac{R^4}{\rho} \right) - \lambda_1 \lambda \rho. \] (5.3)
(5.4)
The rank of the Poisson bracket structure (5.2) is two, i.e., the structure is degenerate. Hence, the reduced system has one degree of freedom. One variable of the set \((p_1, p_2, \rho)\) can be eliminated using the integral (5.4), which is a Casimir function for the structure (5.2).

Traditionally [1], on a two-dimensional level surface of the Casimir function (a symplectic leaf), canonical coordinates \(\{q, p\} = 1\) are introduced. The reduced equations are
\[ \dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}. \]
In contrast to this traditional approach, the local coordinates that we will use account well for the leaf’s geometry but are not canonical. The phase portraits in terms of these coordinates are vivid and illustrative. Our attempts to find a simple and natural set of canonical coordinates have not been successful.

**General properties of motion.** The reduced system (5.2) describes the motion of the vortices relative to the center of the cylinder. Before proceeding to a qualitative analysis of the reduced system, we will indicate some general properties of the absolute motion.

**Proposition 4.** The absolute motion of the cylinder is bounded except maybe for the two cases:

1. \(\lambda = \lambda_1\);
2. \(\lambda = 0\);

**Comment.** Condition (1) corresponds to the case where the circulation around the cylinder is zero, and condition (2) represents the case where the circulation around the cylinder and the vortex is zero.

**Proof.**
Assume the contrary: \(\lambda \neq \lambda_1, \lambda \neq 0\) and the motion of the cylinder is not bounded. Then, according to Prepositions 2 and 3, \(\rho\) is an unbounded function of time and \(\lambda\lambda_1 > 0\). Dividing (5.4) by \(\rho\) yields
\[ \left( \lambda_1 + \frac{p_2}{\rho} - \frac{\lambda_1 R^2}{\rho} \right)^2 + \frac{p_1^2}{\rho^2} = \left( \lambda_1 - \frac{2\lambda_1 R^2 - F}{\rho} \right) = 0 \] (5.5)
Obviously,
\[ \lim_{\rho \to \infty} \left[ \left( \lambda_1 + \frac{p_2}{\rho} \right)^2 + \left( \frac{p_1}{\rho} \right)^2 \right] = \lambda\lambda_1.\]
Therefore, there exist functions \( \alpha(\rho) = o(1) \) and \( \beta(\rho) \) such that 
\[
\lambda_1 + \frac{p_2}{\rho} = \sqrt{\lambda_1^2 + \alpha \cos \beta}, \quad \frac{p_1}{\rho} = \sqrt{\lambda_1^2 + \alpha \sin \beta}.
\]

Upon substitution of these expressions into (5.3), we get
\[
2H = \frac{\rho}{2a} k + \lambda_1 \ln \left( \frac{\rho - R^2 \lambda_1}{\rho^2} \right)
\]

The second term in the right-hand side is of \( O(\ln \rho) \), the factor \( k \) is separated from zero, that is,
\[
k = (\lambda_1 - \sqrt{\lambda_1^2})^2 + \alpha + 2\lambda_1(\sqrt{\lambda_1^2} - \sqrt{\lambda_1^2}2 + \alpha \cos \beta) > k_1 = \text{const} > 0.
\]
This is a contradiction with the fact that \( H \) is constant. \( \blacksquare \)

\textbf{Comment.} One can easily note that for \( \lambda = \lambda_1 \) and \( \lambda = 0 \) an unbounded motion of the cylinder always exists. Therefore the above preposition serves as a criterion of boundedness of the absolute motion. Most likely this criterion remains valid for the case of an arbitrary (finite) number of vortices, but this is not proved yet.

The statement given below provides a nice "geometro-dynamical" insight into the structure of absolute motions of the cylinder.

\textbf{Theorem.} Suppose that \( \lambda \neq 0 \); then for each periodic solution of the reduced system (5.1) there exists a rotating coordinate frame with the origin at the center of vorticity such that with respect to this frame the vortices and the cylinder move along closed curves.

\textbf{Proof.}

It follows from (4.1) that
\[
y x_1 - x y_1 = \frac{p_1}{\lambda}, \quad y y_1 + x x_1 = -\frac{p_2}{\lambda} + \frac{\lambda}{\lambda_1}(R^2 - \rho).
\]
By the assumption of the theorem, the functions in the right-hand side of these equations are periodic functions with period \( T \). Therefore
\[
\begin{pmatrix}
x_1 \\
y_1
\end{pmatrix} = \Phi \cdot \begin{pmatrix}
x \\
y
\end{pmatrix}, \quad \Phi = \begin{pmatrix}
\Phi_1 & \Phi_2 \\
-\Phi_2 & \Phi_1
\end{pmatrix}
\]
(5.7)
where \( \Phi_1 \) and \( \Phi_2 \) are periodic functions with period \( T \). Substituting these expressions into (4.1), we get
\[
\dot{a}x = \lambda y - y G_2 + x G_1, \quad \dot{a}y = -\lambda x + y G_1 + x G_2,
\]
(5.8)
where \( G_1 \) and \( G_2 \) are also \( T \)-periodic functions. Let \( A \) be the matrix for the system of linear differential equations (5.8). It can be easily verified that \( A \cdot \int_0^t A dt = \int_0^t A dt \cdot A \), hence (see, for example, [4]) the fundamental matrix, \( X \), of equations (5.8) can be written as
\[
X = e^{\int_0^t A dt}.
\]
Let us represent \( A \) as a sum
\[
A = B + C,
\]
\[
B = \frac{1}{\lambda} \begin{pmatrix}
0 & \lambda - \langle G_2 \rangle \\
-\lambda + \langle G_2 \rangle & 0
\end{pmatrix}, \quad C = \frac{1}{\lambda} \begin{pmatrix}
G_1 & -G_2 + \langle G_2 \rangle \\
G_2 - \langle G_2 \rangle & -G_1
\end{pmatrix}.
\]
Note that \( \langle G_1 \rangle = 0 \), otherwise equations (5.8) have unbounded solutions which is in a contradiction with Preposition 3. Since the matrices \( B \) and \( C \) commute, we have
\[
X = \begin{pmatrix}
\cos \alpha & \sin \alpha \\
-\sin \alpha & \cos \alpha
\end{pmatrix} \cdot G,
\]
\[
\begin{pmatrix}
\cos \alpha & \sin \alpha \\
-\sin \alpha & \cos \alpha
\end{pmatrix} = \begin{pmatrix}
\frac{\lambda - \langle G_2 \rangle}{\alpha t} & \frac{\lambda - \langle G_2 \rangle}{\alpha t} \\
-\frac{\lambda - \langle G_2 \rangle}{\alpha t} & \frac{\lambda - \langle G_2 \rangle}{\alpha t}
\end{pmatrix} \cdot G.
\]
Here $G$ is a $T$-periodic matrix. Thus, with respect to a coordinate frame (with origin at the center of vorticity) rotating at a rate $(\lambda - (G_2))/a$ the cylinder moves along a closed curve. Let us prove now that the vortex also moves along a closed curve. Let $x^*, y^*$ be the coordinates of the vortex in the fixed coordinate frame. Then $x^* = x_c + x$, $y^* = y_c + y$. It follows from (5.7) that

$$
\begin{pmatrix}
  x^* \\
  y^*
\end{pmatrix} = (\Phi + E) \cdot \begin{pmatrix}
  x \\
  y
\end{pmatrix} =
\begin{pmatrix}
  \cos \frac{\lambda - (G_2)}{a} t \\
  -\sin \frac{\lambda - (G_2)}{a} t
\end{pmatrix} \cdot \begin{pmatrix}
  \frac{\lambda - (G_2)}{a} t \\
  \frac{\lambda - (G_2)}{a} t
\end{pmatrix} \cdot (\Phi + E) \cdot \begin{pmatrix}
  x(0) \\
  y(0)
\end{pmatrix}
$$

Here $E$ is the identity matrix, and the matrix $(\Phi + E)G$ is $T$-periodic.

---

Fig. 3. a) Trajectories of the cylinder (solid line) and the vortex (here the two frequencies of the motion are rationally related). $a = 6, \Gamma = -3, \Gamma_1 = 1, R = 1, F = 2$; the initial conditions (relative to the center of the cylinder) are $v_1(0) = 0.24, v_2(0) = 0, x(0) = 0, y(0) = 1.31$; b) The trajectories shown in diagram a in the frame of reference rotating at a rate 0.029 (the origin, $O_+$, is at the center of vorticity).

An absolute motion of the cylinder (solid line) and the vortex (dashed line) is shown in Fig. 3 a. In Fig. 3 b this motion is shown in a rotating frame of reference whose origin, $O_+$, is at the center of vorticity. The value of the physical parameters for this motion are $a = 6, \Gamma = -3, \Gamma_1 = 1, R = 1, F = 2$ and the initial conditions are $v_1(0) = 0.24, v_2(0) = 0, x(0) = 0, y(0) = 1.31$.

**Qualitative analysis of the reduced system.** The geometry of the symplectic leaf (5.4) of the Poisson bracket structure (5.2) (the phase space of the reduced system) is governed by $\lambda \lambda_1$. The symplectic leaf is compact if $\lambda \lambda_1 < 0$ and non-compact if either $\lambda \lambda_1 > 0$ or $\lambda = 0$. Let us consider these three cases.

1. **Compact case** ($\lambda \lambda_1 < 0$). From (5.4) it follows that

$$
(\lambda_1 \rho + p_2 - \lambda_1 R^2)^2 + p_1^2 + \left(\sqrt{-\lambda \lambda_1} \rho + K\right)^2 = K^2, \quad K = \frac{2 \lambda_1 R^2 - F}{2\sqrt{-\lambda \lambda_1}}
$$

The symplectic leaf is diffeomorphic to a two-dimensional sphere. For real motions $F \geq 2\lambda_1 R^2 (\lambda_1 - \lambda)$. Since the phase space is compact, $\rho$ is a bounded function, meaning that the distance between the vortex and the cylinder cannot grow infinitely. Moreover, in view of Preposition 3, the absolute motion of the cylinder also is bounded.
On the leaf we introduce coordinates \( \varphi \) and \( \psi \) by the formulae

\[
\begin{align*}
\lambda_1 \rho + p_2 - \lambda_1 R^2 &= K \cos \varphi \cos \psi, \quad p_1 = K \sin \varphi \cos \psi, \\
\rho \sqrt{-\lambda_1} + K &= K \sin \psi, \quad \psi \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right), \quad \varphi \in [-\pi, \pi].
\end{align*}
\]

(5.10)

**Comment.** For a fixed value of \( K \), the coordinate \( \psi \) satisfies the relation \( \rho = \frac{K \left( \sin \psi - 1 \right)}{\sqrt{-\lambda_1}} > R^2 \).

To find stationary solutions of the reduced system, consider the differential equations in \( p_1, p_2 \) and \( \rho \):

\[
\dot{p}_1 = \{p_1, H\} = \frac{\lambda_1 R^4 \lambda \alpha \rho - 2R^2 \lambda_1 \lambda \alpha \rho^2 - \lambda_1^2 \alpha \rho^3 + \lambda_1^2 R^2 \alpha \rho^2 + \lambda_1 \lambda \alpha \rho^3}{\rho^2 a(-\rho + R^2)}
\]

\[
+ \frac{(-R^4 \lambda_1 \rho + \lambda \alpha \rho^2 + R^6 \lambda_1 - \lambda_1 R^2 \rho^2 - \lambda R^2 \alpha \rho + \lambda R^2 \rho^2 - \lambda_1 \rho \alpha \rho^2 + \lambda_1 \rho^3 - \lambda \alpha \rho^3)p_2}{\rho^2 a(-\rho + R^2)}
\]

\[
+ \frac{(\rho^2 - R^4)p_2 + (\rho^2 - 2R^2 \rho + R^4)p_1^2}{\rho^2 a(-\rho + R^2)},
\]

(5.11)

\[\dot{p}_2 = \{p_2, H\} = \left( \frac{\lambda \rho^3 - \lambda_1 \rho^3 + \lambda R^2 \alpha \rho - R^6 \lambda_1 + \lambda_1 \alpha \rho^2 - \lambda \alpha \rho^2 - \lambda R^2 \rho^2 + \lambda_1 R^2 \rho^2 + R^4 \lambda_1 \rho \right) p_1
\]

\[
+ \frac{(-2R^2 \rho + 2R^4)p_2 p_1}{\rho^2 a(-\rho + R^2)}
\]

\[
\dot{\rho} = \{\rho, H\} = \frac{(-2 \rho + 2R^2)p_1}{\rho a \rho}.
\]

The last equation implies \( p_1 = 0 \) and the right-hand side of the second equation becomes zero. Therefore, on the phase portrait, all stationary solutions belong to the lines \( \psi = \pm \frac{\pi}{2}, \varphi = 0 \) and \( \varphi = \pm \pi \).

To determine \( p_2 \) and \( \rho \) we should use (5.9) and equate to zero the right-hand side of the first equation.

The number of stationary solutions depends on \( F \).

Equations (5.11) remain unaltered under the transformation \( (p_1, p_2, \rho, t, \lambda, \lambda_1) \to (-p_1, -p_2, \rho, -t, -\lambda, -\lambda_1) \). In view of (5.10), this transformation implies the following change of variables and physical parameters \( (\varphi, \psi, \lambda, \lambda_1) \to (\varphi + \pi, \psi, -\lambda, -\lambda_1) \). Therefore, a change in sign of the circulation and the vortex strength results in a shift of the phase curves along the \( \varphi \)-axis by \( \pi \). Thus, with no loss of generality, we can assume that \( \lambda < 0 \) and \( \lambda_1 > 0 \).

Phase portraits for various values of \( F \) (the value of the other parameters are fixed \( a = 20, \Gamma = -1, \Gamma_1 = 0.5 \) and \( R = 1 \)) are shown in Figs. 4a, 4b and 4c. The “non-physical” area \( (\rho \leq R^2) \) is shown in grey.

The solution curves of the reduced equations are the level curves of (5.3) on the surface (5.4). The level curves are closed, hence the solutions of the reduced equations are periodic.

The fixed points on the phase portraits (stationary solutions) represent a motion in which the cylinder and the vortex move along concentric circles whose centers are at the center of vorticity. For example, the trajectories of the vortex and the cylinder that correspond to the elliptic point in Fig. 4 are shown in Fig. 5. With an increase in \( F \) two more fixed points appear, the motion corresponding to these points is qualitatively shown in Fig. 6.

2. **Non-compact case** (\( \lambda_1 > 0 \)). It follows from (5.5) that

\[
\left( \lambda_1 \rho + p_2 - \lambda_1 R^2 \right)^2 + p_1^2 - \left( \sqrt{\lambda_1 \rho} - K \right)^2 = -K^2, \quad K = \frac{2 \lambda_1^2 R^2 - F}{2 \sqrt{\lambda_1}}.
\]

(5.12)
Fig. 4. Phase portraits of the reduced system in the compact case ($\lambda \lambda_1 < 0$): $a = 20, \Gamma = -1, \Gamma_1 = 0.5, R = 1$.

Symplectic leaves are diffeomorphic to a hyperboloid of two sheets. For real motions $F > 2\lambda_1 R^2(\lambda_1 - \lambda)$. On symplectic leaves we introduce local coordinates:

$$\lambda_1 \rho + p_2 - \lambda_1 R^2 = K \cosh \varphi \sinh \psi, \quad p_1 = K \sinh \varphi,$$

$$\sqrt{\lambda \lambda_1} - K = -K \cosh \psi \cosh \varphi,$$

(5.13)

Arguing as in the compact case, we can assume that $\lambda > 0$ and $\lambda_1 > 0$. In view of (5.11) and (5.13), on the phase portrait, all fixed points lie on the axis $\varphi = 0$. It is interesting to note that, unlike the compact case, the topology of the phase portrait is determined by the sign of the difference $\lambda - \lambda_1$ and does not change qualitatively as the constant $F$ varies. Typical phase portraits are given in Fig. 7a ($\lambda = \lambda_1$), 7b ($\lambda < \lambda_1$) and 7c ($\lambda > \lambda_1$).

The phase curves are closed, hence the solutions of the reduced equations are periodic functions of time. The only exception is the case $\lambda = \lambda_1$ (Fig. 7). As shown above, only in this case the cylinder may have an unbounded motion (Preposition 4).
3. **Non-compact case** ($\lambda = 0$). In the system of equations (1.1), the equations governing the motion of the vortex are uncoupled from the first equation. Indeed, it follows from (3.1) that

\[ av_1 = -\lambda_1 y_1 f(x_1, y_1) + c_1, \quad av_2 = \lambda_1 x_1 f(x_1, y_1) + c_2, \]

where \( f(x_1, y_1) = -1 + R^2/(x_1^2 + y_1^2) \) and \( c_1, c_2 \) are arbitrary constants. Substituting these expressions into the first equation (1.1), we obtain a closed system of equations in the unknowns \( x_1 \) and \( y_1 \). The solution curves of this system coincide with level curves of the integral (5.3). It can be shown that the level curves are closed, hence \( x_1 \) and \( y_1 \) are periodic functions of time. The evolution of the center of the cylinder is governed by the equations

\[ x = \frac{1}{a} (-\lambda_1 (y_1 f(x_1, y_1)) + c_1) t + g_1(t), \]
\[ y = \frac{1}{a} (\lambda_1 (x_1 f(x_1, y_1)) + c_2) t + g_2(t), \]

where \( g_1(t), g_2(t) \) are periodic functions. Thus, there exists a uniformly moving coordinate system in which the orbits of the cylinder and the vortex are closed.
6. The case of two vortices

Suppose now that \( n = 2 \), i.e., we are going to consider the system of a cylinder interacting dynamically with two vortices. By analogy with the case \( n = 1 \), we use the integral (3.3) to reduce the number of degrees of freedom by one. As before, we take some quantities invariant with respect to rotations about the cylinder’s center as the new variables, namely,

\[
p_1 = a(x_1 v_1 + y_1 v_2), \quad p_2 = a(x_1 v_2 - y_1 v_1), \quad p_3 = x_1 x_2 + y_1 y_2, \quad p_4 = x_1 y_2 - x_2 y_1,
\]

\[
r_1 = x_1^2 + y_1^2, \quad r_2 = x_2^2 + y_2^2.
\]

The Poisson brackets for these variables are as follows:

\[
\{p_1, p_2\} = \frac{(p_2^2 + p_1^2) r_2^2 - 2 \lambda_1 R^2 p_2 r_2^2 + (R^4 - r_1^2) \lambda_1^2 r_2^2 + r_1^2 \lambda_1 r_2^2 + r_2^2 \lambda_1 \lambda_2 (R^4 - r_2^2)}{\lambda_1 r_1 r_2^2},
\]

\[
\{p_1, p_3\} = \frac{r_2^2 (p_3 p_2 - p_4 p_1) + \lambda_1 (- (R^2 - r_1) r_2^2 p_3 + 2 R^2 p_2^2 r_1 - R^2 r_1^2 r_2 + r_1^2 r_2^2)}{\lambda_1 r_1 r_2^2},
\]

\[
\{p_1, p_4\} = \frac{r_2^2 (p_1 p_3 + p_2 p_4) - \lambda_1 p_4 (2 R^2 p_3 r_1 + (R^4 - r_1^2) r_2^2)}{\lambda_1 r_1 r_2^2},
\]

\[
\{p_1, r_1\} = \frac{2 p_2 - 2 (R^2 - r_1) \lambda_1}{\lambda_1}, \quad \{p_1, r_2\} = - \frac{2 p_3 (R^2 - r_2)}{r_2},
\]

\[
\{p_2, p_3\} = - \frac{r_2^2 (p_3 p_2 + p_4 p_3) - \lambda_1 p_4 (2 R^2 p_3 r_1 - (R^2 + r_1^2) r_2^2)}{\lambda_1 r_1 r_2^2},
\]

\[
\{p_2, p_4\} = \frac{r_2^2 (p_3 p_2 - p_4 p_3) + \lambda_1 (- (R^2 + r_1) r_2^2 p_3 - 2 R^2 p_2^2 r_1 + R^2 r_1^2 r_2 + r_1^2 r_2^2)}{\lambda_1 r_1 r_2^2},
\]

\[
\{p_2, r_1\} = \frac{2 p_1}{\lambda_1}, \quad \{p_2, r_2\} = - \frac{2 p_4 (R^2 - r_2)}{r_2}, \quad \{p_3, p_4\} = \frac{\lambda_2 r_2 - \lambda_1 r_1}{\lambda_2 \lambda_1},
\]

\[
\{p_3, r_1\} = \frac{2 p_4}{\lambda_1}, \quad \{p_3, r_2\} = - \frac{2 p_4}{\lambda_2}, \quad \{p_4, r_1\} = \frac{2 p_3}{\lambda_1}, \quad \{p_4, r_2\} = \frac{2 p_3}{\lambda_2}, \quad \{r_1, r_2\} = 0.
\]

This Poisson bracket structure is degenerate, and its rank is four. The integral (3.3) and the obvious relation \( p_3^2 + p_4^2 = r_1 r_2 \) are Casimir functions for this structure. For the reduced system to be integrable, one more first integral is needed.

To explore the reduced system numerically, we have used the Poincaré surface-of-section technique. The variables \( r_2, p_3, p_4 \) can be considered local coordinates on a three-dimensional manifold on which the integrals (1.4) and (3.3) are constant and the equation \( p_3^2 + p_4^2 = r_1 r_2 \) is fulfilled; \( p_4 \) is the cross variable. Two Poincaré surface-of-section plots are given in Fig. ???. The chaotic behavior of solutions proves that in the general case an additional first integral does not exist.

7. Conclusion

Very often equations of motion that have not been derived within the framework of Lagrangian formalism (i.e., using the calculus of variations) are not Hamiltonian (even though the energy may be a conserved quantity for these equations). For example, for equations of the nonholonomic mechanics there are dynamical obstacles preventing the existence of a Poisson bracket structure [3]. Since the equations of motion (1.1) are Hamiltonian, they have the generic features of Hamiltonian dynamical systems: for any value of parameters there are no attractors (e.g., strange attractors) in the phase space; at the same time, there are invariant KAM tori separated with stochastic layers. In this paper,
the chaotic system of a cylinder and two vortices has been touched briefly. It seems that it would be interesting to explore some particular motions of this system (both regular and chaotic) in greater detail.

Mention should be made of the paper [8] where a modification of the famous Bjerknes problem, the system of two dynamically interacting 2D cylinders, is considered. The equations of motion for this system are not integrable.

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References