Chaos in a Restricted Problem of Rotation of a Rigid Body with a Fixed Point

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Abstract—In this paper, we consider the transition to chaos in the phase portrait of a restricted problem of rotation of a rigid body with a fixed point. Two interrelated mechanisms responsible for chaotization are indicated: (1) the growth of the homoclinic structure and (2) the development of cascades of period doubling bifurcations. On the zero level of the area integral, an adiabatic behavior of the system (as the energy tends to zero) is noted. Meander tori induced by the break of the torsion property of the mapping are found.

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1. INTRODUCTION

The Euler–Poisson equations that describe the motion of a rigid body about a fixed point in the homogeneous gravitational field have the form

$$\begin{align}
\dot{\mathbf{I}} \omega + \mathbf{I} \omega \times \mathbf{I} \omega &= \mu \mathbf{r} \times \gamma, \\
\dot{\gamma} &= \gamma \times \omega,
\end{align}$$

where $\omega = (\omega_1, \omega_2, \omega_3)$, $\mathbf{r} = (r_1, r_2, r_3)$, and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ are the components of the angular velocity, the position vector of the center of mass, and the unit vector of the vertical direction in the frame of principal axes of inertia $(e_1, e_2, e_3)$ that are attached to the rigid body and pass through the fixed point, $\mathbf{I} = \text{diag}(I_1, I_2, I_3)$ is the tensor of inertia with respect to the fixed point in the same coordinate frame, and $\mu = mg$ is the weight of the body (see Fig. 1).

![Fig. 1. A rigid body with a fixed point in the gravitational field.](image-url)
Equations (1) possess three first integrals

\[ H = \frac{1}{2}(\omega, I\omega) - \mu(r, \gamma), \quad F_1 = (I\omega, \gamma), \quad F_2 = \gamma^2, \]

where \( H \) is the total energy of the body, the integral \( F_1 \) is connected with the symmetry with respect to rotations about the immovable vertical axis and is called the area integral, and \( F_2 \) is the geometric integral whose value for real motions of the rigid body is equal to unity, \( F_2 = \gamma^2 = 1 \).

Consider the problem on the motion of a dynamically symmetric body under the following assumptions: \( I_1 = I_2 = 1, I_3 = \delta, r = (0, \delta, 0) \). For sufficiently small \( \delta < 2 \), the moment of inertia satisfies the triangle inequalities and, therefore, some real mass distribution corresponds to the configuration chosen. In this case, the equations of motion take the form

\[ \dot{\omega}_1 = (1 - \delta)\omega_2\omega_3 - \delta\gamma_3, \quad \omega_2 = (\delta - 1)\omega_1\omega_3, \quad \dot{\omega}_3 = \gamma_1. \]

Consider the limit case of this problem as \( \delta \to 0 \), where the body degenerates into a segment of a straight line. This limit transition is completely similar to the transition to the restricted three-body problem in celestial mechanics. In this case, the moment of inertia and the moment of the gravitational force with respect to the axis of dynamical symmetry simultaneously tend to zero. Thus, in the limit, we obtain a nontrivial equation for the proper rotation, which will be considered below. This limit transition was first proposed by V. V. Kozlov and D. V. Treschev in [1]. A similar limit transition in a more general case of a dynamical nonsymmetric body was considered by A. A. Burov in [2].

After the limit transition as \( \delta \to 0 \), Eqs. (3) take the form

\[ \dot{\omega}_1 = \omega_2\omega_3, \quad \dot{\omega}_2 = -\omega_1\omega_3, \quad \dot{\omega}_3 = \gamma_1. \]

The integrals for system (4) are obtained from the integrals (2) of the initial problem by the limit transition and have the form

\[ \omega_1^2 + \omega_2^2 = h, \quad \omega_1\gamma_1 + \omega_2\gamma_2 = c, \quad \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1. \]

We reduce Eqs. (4) to the common level of integrals (5) (see [1]). For this, take as the variables \( \xi, \eta, \) and \( \gamma_3 \), which are related with \( \omega_1, \omega_2, \) and \( \omega_3 \) by the formulas

\[ \omega_1 = \sqrt{2h}\sin\xi, \quad \omega_2 = \sqrt{2h}\cos\xi, \quad \omega_3 = \eta. \]

The equation for \( \gamma_3 \) separates and integrating it we obtain \( \gamma_3 = A\cos(\sqrt{2ht}) \), where \( A \) depends on the values of integrals (5). Substituting the solution obtained in Eqs. (16) for \( \dot{\gamma} \) and integrals (5), we have

\[ \gamma_1 = \frac{c}{\sqrt{2h}}\sin\xi - \sqrt{1 - \frac{c^2}{2h}}\sin(\sqrt{2ht})\cos\xi, \]

\[ \gamma_2 = \frac{c}{\sqrt{2h}}\cos\xi + \sqrt{1 - \frac{c^2}{2h}}\sin(\sqrt{2ht})\sin\xi, \]

\[ \gamma_3 = \sqrt{1 - \frac{c^2}{2h}}\cos(\sqrt{2ht}). \]

It is easy to show that for all motions of the rigid body, the radicands in (7) are nonnegative. Substituting (6) and (7) in Eqs. (4) we obtain the equations for the variables \( \xi, \eta \):

\[ \dot{\xi} = \eta, \]

\[ \dot{\eta} = \frac{c}{\sqrt{2h}}\sin\xi - \sqrt{1 - \frac{c^2}{2h}}\sin(\sqrt{2ht})\cos\xi. \]

Equations (8) can be represented in the Hamiltonian form

\[ \frac{\dot{\xi}}{2} = \frac{\partial H}{\partial t}, \quad \dot{\eta} = -\frac{\partial H}{\partial \xi}, \]

\[ H = \frac{\gamma^2}{2} + \frac{c}{\sqrt{2h}}\cos\xi + \sqrt{1 - \frac{c^2}{2h}}\sin(\sqrt{2ht})\sin\xi. \]
Therefore, the considered limit case of the problem on the rotation of a heavy rigid body about a fixed point is reduced to a Hamiltonian system with one and a half degrees of freedom.

Now we consider properties of the absolute motions in the restricted problem on the motion of a rigid body. For this, consider the quadrature for the precession angle

\[ \dot{\psi} = \frac{\omega_1 \gamma_1 + \omega_2 \gamma_2}{\gamma_1^2 + \gamma_2^2}. \]  

(10)

Substituting (5) and (7) in (10) and integrating the relation obtained in time, we have

\[ \psi(t) = \begin{cases} \arctan \left( \frac{\sqrt{2h}}{c} \tan(\sqrt{2ht}) \right) + \psi(0), & c \neq 0, \\ \psi(0), & c = 0. \end{cases} \]  

(11)

In the limit case considered, the body is degenerated into a segment of a straight line, therefore, its motions in the immovable coordinate frame are described only by the apex \( e_3 \), which is parallel to the third axis of inertia. The dependence of \( e_3 \) on the Euler angles has the form

\[ e_3 = (\sin \theta \sin \psi, -\sin \theta \cos \psi, \cos \theta). \]  

(12)

Taking the relation \( \cos \theta = \gamma_3 \) into account and applying (11), we see that for any values of the integrals and initial conditions for Eqs. (9), the motion of the rod is periodic with frequency \( \sqrt{2h} \).

In particular, for \( c = 0 \), the precession is absent and any motion of the body is a uniform rotation in the vertical plane. Therefore, in the limit state, the equations of motion are divided into two parts. The first part is connected with the absolute motion of the body and can be explicitly integrated. The second part is connected with the proper rotation of the body and, in general, is not related to the real motion of the body since actually describes the rotation of an infinitesimally thin rod about its axis. An interesting fact holds: even in the case of a chaotic proper rotation, the body moves regularly and periodically in the immovable coordinate frame. Moreover, owing to the limit transition in the restricted problem, we avoid problems related with the topology of isoenergetic levels and the choice of the global transversal Poincaré section. We also note that the main properties of the evolution of chaos described in the present paper also appear in the general Euler–Poisson equations (see [3]).

Note that system (9) can also be considered as the problem on the mathematical pendulum with periodic in time perturbation of special form. Indeed, setting \( \nu = \sqrt{1 - c^2 / 2h} \) and assuming that \( \nu \) is a small parameter, we obtain

\[ \mathcal{H} = \mathcal{H}_0 + \nu \mathcal{H}_1 + o(\nu), \quad \mathcal{H}_0 = \frac{\eta^2}{2} + \cos \xi, \quad \mathcal{H}_1 = \sin(\sqrt{2ht}) \sin \xi. \]  

(13)

Hence, for \( \nu = 0 \), system (9) is integrable. Its analytic nonintegrability for \( \nu \neq 0 \) was proved in [1] for the case where \( c \neq 0 \) and in [4] for the case where \( c = 0 \) by numerical construction of splitting separatrices for an unstable periodic solution. Moreover, in [5] the Kovalevskaya exponents are calculated and hence the algebraic nonintegrability of this problem is proved. We first consider the chaotization in the problem considered if \( c = 0 \).

2. CHAOTIZATION IN THE CASE WHERE \( c = 0 \)

Consider the evolution of the phase portrait of system (9) for \( c = 0 \) and the change of the total energy \( h \) of the body (see (5)) from \( +\infty \) to 0. For this, we study the mapping for a period of the perturbation on the plane \( (\xi \mod 2\pi, \eta) \) for Eqs. (8).

The mapping considered is symmetric with respect to the changes \((\eta \rightarrow -\eta, \xi \rightarrow \pi - \xi)\) and \((\xi \rightarrow -\xi, t \rightarrow -t)\). This allows us to restrict ourselves to the study of the dynamics of the system in the domain \( \eta \geq 0, \xi \in (0, \pi) \); however, for the clearness, we present full phase portraits. We also note that the most interested is the behavior of the system for sufficiently small \( \eta \) since for large \( \eta \), terms depending on \( \xi \) and \( t \) in the Hamiltonian (9) can be neglected and the motion becomes close to integrable. On the phase portrait this emerges as follows: as \( \eta \) increases, the distinction of invariant curves from horizontal straight lines vanishes.

Note that there exist two supplemental mechanisms of the chaotization of the phase portrait:
transversal intersections of unstable invariant manifolds — separatrices (and appearing in this case homoclinic structure of the mapping and the Smale horseshoe). This mechanism is responsible for the appearing of a chaotic layer near unstable periodic solutions.

cascades of period doubling bifurcations. After the passing of a cascade of period doubling bifurcations, the phase portrait is characterized by the presence of trajectories with arbitrarily large periods. Chaos caused by such behavior can be called local since it is connected with concrete periodic solutions. The global chaotization of the phase portrait near the cascade is caused by the fact that owing to the cascade, domains of the regularity split into smaller parts around which chaos appears by the first mechanism of chaotization.

As we will see below, in the problem considered, the combined action of both mechanisms leads to the full chaotization of the phase portrait.

Now we consider in detail all stages of the chaotization of the phase portrait of the problem as the energy decreases.

Splitting of resonance tori. If $h = +\infty$, system (9) is integrable and the corresponding phase portrait on the plane $(\xi, \eta)$ has the form of horizontal straight lines; to each of these lines, its own frequency of rotation in $\xi$ corresponds. As the energy decreases, invariant curves for which the frequency of rotation in $\xi$ is commensurable with the frequency of the constraining force $\sqrt{2h}$ (so-called resonance tori) split. In Fig. 2, the corresponding phase portrait of the system for sufficiently large but finite energy is presented. As is seen from the figure, the splitting of resonance tori for this value of $h$ is sufficiently significant; however, the phase portrait is still close to the integrable case. In this case, resonances (periodic solution) of the first order are determinative for the phase portrait. All periodic solutions that are materialized as the perturbation increases (the energy decreases), are numerated in Fig. 2. Resonances of higher orders for a given value of energy do not materialize, i.e., split weakly and practically do not differ from usual tori.

![Fig. 2. The phase portrait for $c = 0$ and $h = 8$.](image)

Remark 1. The fixed points in Fig. 2 labelled by 1 and 2 correspond to permanent rotations on the body about the axes $Ox$ and $Oy$; moreover, to different points with the same number, the same two rotations in the opposite directions correspond. The fixed points labelled by 3 and 4 correspond to periodic (in the absolute space) solutions which we call choreographies following [6]. During the motion along the corresponding periodic solutions, the apexes $Ox$ and $Oy$ run over closed figure-eight curves and the apex $Oz$ moves along a circle in the vertical plane.

Homoclinic structure of the mapping. As the energy decreases, the separatrices are split near unstable periodic solutions and narrow chaotic layers appear around stable periodic solutions of different orders (see Fig. 3). A method of constructing separatrices can be found in [7].
Remark 2. It is interesting that the upper pair of separatrices of a periodic solution corresponding to an unstable choreography practically do not split as compared with the lower pair (see Fig. 3 upper right). Note that this situation occurs in almost all range of $h$ (except for sufficiently small values).

In further decreasing of the energy, tori separating chaotic layers break and chaotic layers merges. Moreover, the separatrices corresponding to different fixed points transversally intersect. After the merging of all chaotic layers, one chaotic layer appears; it is formed by the net of intersections of separatrices of different orders. The corresponding phase portrait and the intersections of the separatrices of the unstable fixed points of orders 1, 2, 3, 4, 5, and 7 are presented in Fig. 4.

Remark 3. The closure theorem [8] and the existence of intermediate intersections of higher-order separatrices imply that separatrices corresponding to the upper and lower first-order fixed points in Fig. 4 also intersect and their closures coincide. However, near the critical value of $h$ at which the last invariant torus that separates chaotic layers breaks, this intersection can happen at very large distance (in the sense of number of iterations) from fixed points. Therefore, exponentially large time is required for finding the first intersection of separatrices.

Cascades of period doubling bifurcations. On the next stage of chaotization, as the energy decreases, domains of regularity breaks through cascades of period doubling bifurcations. These cascades are infinite sequences of period doubling (or multiplying) bifurcations when the parameter of the system varies in finite limits. Different types of bifurcations of periodic solutions and cascades of bifurcations are described in more detail in the Appendix. After passing cascades, a sufficiently homogeneous chaotic layer near the origin appears (see Fig. 5).

Note that, as a rule, some previous bifurcations precede cascades. For illustration, we present the scheme of bifurcations in the case where the energy decrease for two most important stable first-order periodic solutions.

- Periodic solution 1 (Fig. 2).
  1. a period doubling bifurcation;
  2. a pitchfork bifurcation;
  3. a cascade of period doubling bifurcation.

- Periodic solution 3 (Fig. 2).
  1. a double period doubling bifurcation in which the fixed point keeps its stability type, and two pairs of stable and two pairs of unstable solutions of double period appear from it;
  2. a cascade of period doubling bifurcation of the appeared pair of stable two-order periodic solutions;
  3. a pitchfork bifurcation of the main solution;
  4. a cascade of period doubling bifurcation.
Fig. 4. The phase portrait and intersections of separatrices of orders 1, 2, 3, 4, 5, and 7 for $c = 0$ and $h = 2$. (a) and (b) the phase portrait and its magnification in the domain of construction of separatrices; (c)–(i) consecutive intersections of separatrices. Numbers near fixed points denote their periods.

Fig. 5. The phase portrait and separatrices of first-order fixed points for $c = 0$ and $h = 1$.

The third periodic solution lies in the symmetry plane $\xi = \pi$. It was shown in [9] that for an invertible mapping, under the bifurcation of a periodic solution lying in the symmetry plane, new periodic solutions appeared intersect this plane at two points. In addition, in the continuation by the parameter, these solutions remain in the symmetry plane. Therefore, in the study of doubling bifurcations of the third periodic solution, it suffices to observe only periodic solutions appeared that lie on the axis $\xi = \pi$. The corresponding projection of the doubling bifurcation tree on the plane $(h, \eta)$ is shown in Fig. 6. The constants of the scaling transformation by the parameter $h$ for the first six doubling bifurcations are presented in Table 1. It is seen from the table that the values of the coefficients of the scaling transformation sufficiently quickly converge to the Feigenbaum constant $\delta = 8.721 \ldots$. 
Fig. 6. The projection of the tree of period doubling bifurcations, which starts from absolute choreographies, on the plane $(h, \eta)$.

Table 1. Coefficients of the scaling transformation for the bifurcation tree from Fig. 6

<table>
<thead>
<tr>
<th>Number of bifurcation</th>
<th>$h_n$</th>
<th>$\delta_n$</th>
<th>Number of bifurcation</th>
<th>$h_n$</th>
<th>$\delta_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.287733199</td>
<td>2</td>
<td>0.734936403</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.282333473</td>
<td>3</td>
<td>0.721745207</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.281711468</td>
<td>8.681161784</td>
<td>4.720016837 7.632159557</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.281640344</td>
<td>8.745399858</td>
<td>5.719817826 8.684798818</td>
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<td></td>
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<tr>
<td>6</td>
<td>0.28163219</td>
<td>8.721983486</td>
<td>6.719795025 8.728028583</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

As was noted above, after cascades of period doubling bifurcations, the phase portrait of the system for small $\eta$ becomes practically totally chaotic. However, for some (sufficiently small) value of the energy, on the axis $\xi = \pi$, a stable first-order periodic solution appears again. If the energy further decreases, this solution undergoes a pitchfork bifurcation and then the regularity domain appeared becomes chaotic by another cascade of period doubling.

The case $c = 0$, $h \to 0$. Adiabatic behavior. Consider the evolution of the phase portrait as the energy $h$ tends to zero. For this, we recall that for fixed $h$ and infinitely large $\eta$, system (9) becomes integrable and all its motions are uniform rotations in $\xi$ for fixed $\eta$. Now consider Eqs. (9) for large $\eta$ as a periodic in time perturbation of such integrable system. As a perturbation, the second and third terms in the Hamiltonian (9) serve, and the increasing of the perturbation corresponds to the decreasing of $\eta$.

Introduce the rotation number as follows:

$$n = \frac{1}{2\pi} \int_0^T \xi \, dt,$$

where $T = 2\pi/\sqrt{2h}$ is the period of the perturbation. As usual, the addition of the perturbation leads to the breaking of resonance tori for which the equality $n = p/q$, $p, q \in \mathbb{N}$, holds. In this case, by the Poincaré theorem [10], from such tori, an even number of order-$q$ periodic solutions appear.
Substituting (8) in (14), we obtain for large \( \eta \)

\[
n = \frac{\eta}{\sqrt{2h}} + o(\eta). \tag{15}
\]

As is seen from (15), the rotation number unboundedly grows as \( \eta \) increases. Therefore, for fixed \( h \), in system (9) there exist infinitely many first-order periodic solutions that appear when resonance tori with integer rotation numbers break (\( q = 1 \)).

Consider the evolution of these periodic solutions with a given rotation number \( n \). Equation (15) implies that as \( h \) decreases, periodic solutions with a given rotation number moves toward the origin. For sufficiently small \( \eta \), the terms of the Hamiltonian cannot be considered as a small perturbation. Hence, the further evolution of fixed points can be studied only by applying of computer methods, in particular, by the method of continuation of periodic solutions in a parameter (see the Appendix).

It turns out that when \( h \) further decreases, fixed points appeared from resonance tori continue to approach to the chaotic layer, which appeared after cascades of period doubling bifurcation. At some critical value of \( h \), this layer merges with a narrow chaotic layer near unstable periodic solutions with rotation number \( n \). The domains of regularity near stable periodic solutions with rotation number \( n \) remaining after this become chaotic by cascades of period doubling bifurcation.

The evolution described occurs for first-order fixed points with integer rotation numbers \( n \geq 3 \). Therefore, as the energy \( h \) tends to zero, in the system, there occurs an infinite number of cascades of period doubling bifurcations of first-order fixed points that descend from infinity by \( \eta \) in chaotic layer.

Note that for any arbitrarily small \( c \neq 0 \) and the minimal value of \( h = h_{\text{min}} = c^2/2 \), system (8) is reduced to the equation of mathematical pendulum and is integrable. For \( c = 0 \) and \( h = 0 \), we also obtain an integrable system, but for arbitrarily small \( h > 0 \), a chaotic layer of finite width (not tending to zero) occurs. This behavior is similar to the behavior of systems with adiabatic chaos. Indeed, for \( c = 0 \), the equations of motion with the Hamiltonian \( \mathcal{H} = \frac{p^2}{2} + \sin(\sqrt{2h}t) \sin \xi \) for small \( h \) can be considered as an adiabatic system with slowly varying parameter \( \tau = \sqrt{2h} \); \( \tau = \sqrt{2h} \to 0 \). Methods of the study of such systems are developed in [11]. Note that these methods are closely related with the study of jumps of the adiabatic invariant when passing through separatrices of hyperbolic fixed points (periodic solutions) and the obtaining of explicit criteria for the splitting these separatrices. A distinguishing feature of our problem is the change of stability of main periodic solutions during a period of the perturbation. As a result, the technique developed in [11] becomes inapplicable. However, experiments show that as \( h \to 0 \), an unremovable chaotic zone exists. It is interesting to find the limit (as \( h \to 0 \)) value of \( \eta \), which bounds this chaotic layer.

3. THE CASE WHERE \( c \neq 0 \)

System (9) is two-parametric and, therefore, in the general case, we must study the system on the plane of the parameters \((c, h)\). Here, we restrict ourselves to the study of the concrete value \( c = 1 \), however, as calculations show, for other values of \( c \), common features of the evolution of chaos are preserved.

We briefly describe main stages of the evolution of the phase portrait.

1. As \( h = +\infty \) (as in the case where \( c = 0 \)), system (9) is integrable and the phase portrait is stratified into parallel straight lines.

2. As \( h \) decreases, resonance tori split and the corresponding chaotic layers increase (see Fig. 7). In this case, Eqs. (9) are invariant only with respect to the change \((\xi \to -\xi, t \to -t)\); therefore, the phase portrait is symmetric only with respect to the plane \( \xi = 0 \). This leads to the following: for sufficiently large energies, solutions 1 (Fig. 2) are absent and one of solutions 2 is stable.

3. As \( h \) further decreases, chaotic layers appeared unite into one layer (see Fig. 8). This stage qualitatively do not differ form the case of \( c = 0 \) considered above.
4. Almost all regularity domains in the chaotic layer appeared (as in the case $c = 0$) break by cascades of period doubling bifurcations. The exceptions in the considered case $c = 1$ are the regularity domain around the solution 2 and some small higher-order resonances (see Fig. 9). Note that in the general case (where $c \neq 0$) they also can become chaotic. For example, for sufficiently small $c$, the solution 2 loses stability by a pitchfork bifurcation. However, in what follows, it undergoes a bifurcation and becomes stable.

5. The further decreasing of $h$ leads to the growth of the regularity domain around the solution 2 and the reduction of the chaotic layer (see Fig. 10). This process terminates as $h = h_{\text{min}} = c^2/2$ by the passing to an integrable system (mathematical pendulum). In the passing, so-called meander tori appear; they will be describes in the following section.

4. MEANDER TORI

Consider in details the evolution of the phase portrait near the minimal value of the energy, i.e., near the mathematical pendulum. On the phase portrait, there exists a pair of first-order periodic solutions, stable and unstable. It turns out that as the energy $h$ increases (the perturbation rises) near the stable periodic solution, the torsion property of the mapping does not hold. A result of the violation of torsion is the appearing of meander tori (see [12]) for which the radius cannot be represented as a single-valued function of the angle (see Figs. 11 and 12). Meander tori in the Hill problem in celestial mechanics were discovered in [13].

We briefly describe the mechanism of formation of meander tori. Since the torsion property does not longer hold, the extremum (maximum in our case) of the rotation number does not coincides with the
main periodic solution but lies on some invariant curve (torus). As a result, to different sides of this curve, there exist resonance tori (resonances for brevity) with the same rotation numbers \( n = \frac{p}{q}, p, q \in \mathbb{N} \). As the perturbation increases (\( h \) increases), the maximal value of the rotation number approaches to the value \( p/q \); as a result, the resonances approach one to the other. Moreover, as \( h \) increases, the splitting of the resonances also increases. At some critical value \( h^* \), the resonances merge and the separatrices belonging to different resonances mutually intersect. After this, tori appear that bend stable periodic solutions of both resonances; these tori are called meander tori. If \( h \) further increases, stable periodic solutions of one resonance approach to unstable solutions of the other resonance and then mutually annihilate. The annihilation occurs at the moment when the maximal value of the rotation number becomes \( p/q \).
Fig. 12. Meander tori near the resonances with rotation number \( n = 1/4 \) for \( c = 1 \) and \( h = 0.57362 \).

Note that meander tori exist in a sufficiently narrow interval of energies \( h \in (h^*, h(n_{\text{max}} = p/q)) \). For example, for meander tori shown in Figs. 11 and 12, this interval has order \( 10^{-4} \). Moreover, the width of the meander torus strongly depends on the value of the splitting of initial resonance tori. Therefore, on the phase portrait, the better visible are meander tori appeared from resonances of small order sufficiently far from the integrable case.

5. APPENDIX. METHODS OF STUDY OF MAPPINGS

5.1. Search for Fixed Points of Mappings

Here and in what follows, we consider area-preserving self-mappings \( T(\mathbf{x}), \mathbf{x} \in \mathbb{R}^2 \), of the plane that appear as period mappings for nonautonomous systems with one degree of freedom or as Poincaré mappings for autonomous Hamiltonian systems with two degrees of freedom at fixed energy. The search for a fixed point of a mapping is reduced to the numerical solution of the equation

\[
T(\mathbf{x}) = \mathbf{x}.
\]

If the monodromy matrix \( L = \partial T / \partial \mathbf{x} \) of the mapping in a neighborhood of a fixed point has eigenvalues not equal to 1, then the fixed point is isolated and can be found by the Newton method. The iteration formula for the search for the fixed point has the form

\[
(L - E)(\mathbf{x}_{i+1} - \mathbf{x}_i) = -(T(\mathbf{x}_i) - \mathbf{x}_i).
\]

If the mapping is induced by a phase flow \( \dot{\mathbf{x}} = f(\mathbf{x}) \), then the monodromy matrix can be found by the solution of the equation in variations

\[
\delta \mathbf{x} = \frac{\partial f}{\partial \mathbf{x}}(\mathbf{x})\delta \mathbf{x}.
\]

For this, as initial conditions for \( \delta \mathbf{x} \), we choose small displacements along the coordinate axes \((\delta x_0^1, \delta x_0^2) = (\varepsilon, 0), (0, \varepsilon)\). Integrate the equations of motion and Eqs. (18) from the point \( \mathbf{x}_0 \) to the next intersection with the section plane. The obtained vectors \( \delta \mathbf{x}_i \) divided by \( \varepsilon \) are the columns of the monodromy matrix \( L = \frac{1}{\varepsilon}(\delta x_1, \delta x_2) \).

The stability of a fixed point of a mapping in the linear approximation is defined by the multipliers, i.e., the eigenvalues \( \lambda_{1,2} \) of the monodromy matrix. Since the mapping \( T(\mathbf{x}) \) preserves the area, we have \( \det L = \lambda_1 \lambda_2 = 1 \). Therefore, the eigenvalues are either complex conjugate and lie on the unit circle or real and reciprocally inverse. In the case of complex conjugate eigenvalues, the fixed point is stable (of elliptic type). If the eigenvalues are real, then the fixed point is unstable (of hyperbolic type or hyperbolic type with inversion if \( \lambda_i < 0 \)). The conclusion on the stability of parabolic (\( \lambda_1 = \lambda_2 = 0 \)) and degenerate (\( \lambda_1 = \lambda_2 = 1 \)) fixed points cannot be made in the linear approximation.
5.2. Continuation of Fixed Points of a Mapping by a Parameter

Consider a self-mapping \( T(x, \mu) \) of the plane depending on a parameter. Let \( x_0 \) be a fixed point of the mapping for \( \mu = \mu_0 \). Under a small change of the parameter \( \mu = \mu_0 + \delta \mu \) it turns into a close fixed point whose first approximation \( x_1 = x_0 + \delta x \) is calculated by the equation

\[
(L - E)\delta x = -\frac{\partial T}{\partial \mu}(x_0, \mu_0)\delta \mu.
\]  

The further specification of the position of the fixed point can be obtained by the Newton method. The consecutive application of this procedure allows one to construct a family of fixed points of the mapping as parameter varies.

5.3. Bifurcations of Fixed Points of Mappings

A bifurcation is a point of ramification, appearing, or disappearing of a fixed point as parameters of the system vary. Possible scenarios of bifurcations are restricted by the conservation of the Poincaré index [14]. The Poincaré index of a closed curve \( C \) that does not pass through fixed points of the mapping is the number of revolutions of the vector \( \theta(x) = T(x) - x \) when its endpoint moves along the curve. The Poincaré index of a fixed point is defined as the index of a curve from a neighborhood of the fixed point considered and passing around only it. It is easy to show that the Poincaré index of an elliptic point is equal to +1, of a hyperbolic point is -1, and of hyperbolic point with inversion is +1. The index of an arbitrary contour is equal to the sum of the indexes of fixed points lying inside this contour. By definition, the Poincaré index is integer and must be conserved if the parameters of the system vary, because of the analyticity of the mapping as a function of parameters. Therefore, the sum of the indexes of fixed point in a given domain under the varying if parameters of the system is preserved (in this case, we assume that fixed points do not cross the boundary of the domain). The rule stated strongly restricts possible bifurcations of fixed points. Consider some bifurcations considered in the present paper.

1. **Pitchfork bifurcations.** As the parameter varies, the multiplicators of the fixed point pass through the value \( \lambda_1 = \lambda_2 = 1 \). The type of the point changes from elliptic to hyperbolic and the Poincaré index changes from +1 to -1. As a result, near the initial point, two stable (unstable) points with indexes +1 (-1) appear (disappear).

2. **Period doubling bifurcations.** As the parameter varies, the multiplicators of the fixed point pass through the value \( \lambda_1 = \lambda_2 = -1 \). The type of the point changes from elliptic to hyperbolic with inversion and the Poincaré index is preserved. However, for the mapping \( T^2(x) \), the Poincaré index changes from +1 to -1 and hence, near the initial point, two stable (unstable) second-order fixed points with indexes +1 (-1) appear (disappear).

Note that far from the integrable case, the loss of stability of a fixed point with appearing of a pair of stable fixed point with double period is most typical. Moreover, such bifurcations can form infinite sequences. We describe them in the next section.

5.4. Cascades of Period Doubling Bifurcations

Consider an infinite sequence of period doubling bifurcations that occur for the values of the parameter \( \mu = \mu_i, \ i \in \mathbb{N} \). In the interval \( \mu \in (\mu_i, \mu_{i+1}) \), the mapping \( T(x, \mu) \) has a stable periodic solution with period \( 2^k \). For \( \mu = \mu_{k+1} \) it becomes unstable, and a stable periodic solution of double period \( 2^{k+1} \) appears in its neighborhood. Such a sequence of bifurcations occurs if the parameter changes in finite limits, possesses a universal asymptotic behavior, and is called a period doubling bifurcation. The universality of the behavior is that the exponent of the convergence rate \( \delta_i = \frac{\mu_{i+1} - \mu_i}{\mu_{i+2} - \mu_{i+1}} \) tends, as \( i \) increases, to the universal Feigenbaum constant, which is independent of the form of the mapping. For two-dimensional conservative mappings, the value of the Feigenbaum constant is \( \delta = 8.72109 \ldots \).

Except for the universality property, for two-dimensional mappings, two additional universal factors exist. It turns out that in an appropriate coordinate frame, the structure of periodic trajectories repeats itself with scales \( \alpha = 4.018 \ldots \) and \( \beta = 16.363 \ldots \) along the coordinate axes.
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