Theoretical and experimental investigations of the rolling of a ball on a rotating plane (turntable)

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Abstract
In this work we investigate the motion of a homogeneous ball rolling without slipping on uniformly rotating horizontal and inclined planes under the action of a constant external force supplemented with the moment of rolling friction, which depends linearly on the angular velocity of the ball. We systematise well-known results and supplement them with the stability analysis of partial solutions of the system. We also perform an experimental investigation whose results support the adequacy of the rolling friction model used. Comparison of numerical and experimental results has shown a good qualitative agreement.

Keywords: rolling, rotating surface, tilted turntable, non-holonomic constraint, rolling ball, rolling friction, qualitative analysis

1. Introduction
This paper is concerned with the motion of a homogeneous ball rolling without slipping on a uniformly rotating plane. This is a classical problem studied in non-holonomic mechanics. As the ball moves, a linear inhomogeneous non-holonomic constraint is satisfied. This constraint corresponds to the condition that the velocities of the contacting points on the surface of the ball and on the rotating plane be equal.

Apparently, Earnshaw in 1844 was the first to investigate the dynamics of the non-holonomic system of interest (in the case of a horizontal plane) [1]. He showed that the
trajectories of the centre of the ball relative to a fixed reference frame are circles and that the position of the centre of these circles depends on initial conditions. In this case, the ball moves with constant angular velocity, which depends only on the angular velocity of the surface on which the ball rolls. The same result was obtained later in [2–6] using different approaches to describing the system. A formulation of the problem of a ball rolling on a rotating plane can also be found in the book of Routh [7] ‘Dynamics of a system of rigid bodies’, in which he compared the motion of a homogeneous ball on an absolutely rough and an absolutely smooth plane.

The late 1970s marked a surge of interest in this problem, in particular, which was stimulated especially by the development of experimental methods of observation and processing of experimental data using computers. In [5, 6], it is shown that the addition of a constant external force (including the projection of the gravity force during motion on an inclined plane) leads to a uniform drift of the centre of mass of the ball along the direction perpendicular to that of the external force. The trajectory of motion is a trochoid traced out by a point lying inside or outside a circle rolling in a straight line with constant velocity and is similar to the trajectory of a charged particle in a constant electromagnetic field [8, 9].

In [2], a detailed analysis is made of the motion of a ball on a rotating horizontal plane under the action of an external force proportional to the velocity of the centre of mass of the ball (similar to the influence of an external viscous medium, for example air). It is shown that in this case the trajectory of the centre of mass of the ball relative to a fixed coordinate system is a spiral that convolves to a fixed point (see also [10]).

We also mention the work of Weltner [11, 12], where he took into account the force of rolling friction which acts on the ball and is proportional to the velocity of the sphere relative to the rotating plane. By some transformations (which, in our opinion, are not quite correct) he obtained equations of motion corresponding to twisting spiral motion of the ball at the initial stage of motion.

The aforesaid suggests that various friction models for the system under consideration lead to results that contradict each other. Therefore, the choice of a friction model should be determined by experimental results. But the experimental results obtained earlier in [5, 10, 13, 11] also contradict each other.

For example, in [11] Weltner demonstrated twisting spiral motion at the initial stage of motion, which was not obtained in our experiments. The authors of [13, 10] showed a qualitative agreement between experimental trajectories and theoretical trajectories obtained under the assumption of a non-holonomic motion model supplemented with the moment of rolling friction. In this case, at the initial stage of motion, the centre of mass of the ball is displaced to the centre (axis) of rotation of the plane, and then the centre of mass of the ball continues to move in an untwisting spiral. In [5], it is pointed out that the trajectories greatly depend on initial conditions, but no trajectory obtained is presented. In the earlier papers [2, 4], only experimental results are mentioned, but the experiment itself is not described.

We also mention the papers [3, 14], which analyse, within the framework of the non-holonomic model, the motion of a ball crossing a rotating circular table. The ball leaves the plate along a trajectory that exactly coincides with the continuation of its original rectilinear trajectory. This elementary, but non-trivial experiment is demonstrated in the Franklin Museum and in the research centre ANAIS (Centre de Culture Scientifique, Technique et Industrielle, Nice).

In this work, we systematisate the well-known results and supplement them with a qualitative stability analysis of partial solutions. We also carry out an experimental investigation of motion on an inclined and a horizontal plane using high-speed cameras and modern data processing means. The results obtained confirm the adequacy of the non-holonomic model
supplemented with the moment of rolling friction, which depends linearly on the angular velocity. Comparison between the numerical and experimental results has allowed us to determine the values of the coefficient of proportionality of the moment of rolling friction and the angular velocity for the pair of materials used.

It is noteworthy that during the teaching process the motion of a ball along spiral trajectories on a rotating plane can be demonstrated quite well even without using expensive laboratory equipment. This can be done, for example, by using a vinyl record player and a ping-pong ball. The observed motion is non-trivial and finds great interest among students. It will also be shown that the dynamics of the ball on the rotating plane under the action of a constant external force and friction torque is described by a linear system of differential equations, which is investigated by standard methods of linear algebra and the theory of ordinary differential equations. The above-mentioned advantages of the problem considered make it indispensable in the study of non-holonomic mechanics.

During the preparation of this work, a paper \cite{15} was published in which the motion of a sphere and a disk on a horizontal rotating plane is investigated. The authors of \cite{15} determine numerically and experimentally the instant of transition from non-holonomic rolling to motion with slipping at large angular velocities of rotation of the plane. The results of the study \cite{15} supplement our results and confirm the correctness of the friction model used in our work.

\section*{2. Equations of motion of a ball rolling on a rotating plane}

Consider the motion of a homogeneous ball with mass \(m\) and radius \(R\) rolling without slipping on an inclined absolutely rough plane. The plane rotates with constant angular velocity \(\Omega\) about the axis aligned with the external normal to the plane (figure 1).

To describe the dynamics of the ball, we define a fixed (inertial) coordinate system \(Oxyz\) with unit vectors \(\alpha, \beta,\) and \(\gamma\) in such a way that the plane \(Oxy\) coincides with the plane of rolling of the ball, the unit vector \(\beta\) is directed along the largest slope, the unit vector \(\gamma\) is directed along the external normal to the plane of motion. The vector of free-fall acceleration in the chosen coordinate system has the form \(g = (0, g \sin \delta, -g \cos \delta)\), where \(\delta\) is the angle of inclination of the plane of motion (figure 1).

The position and configuration of the system are completely defined by the coordinates of the centre of the sphere \(r = (x, y, 0) \in \mathbb{R}^2\) and by the orthogonal matrix of rotation of the ball \(S \in SO(3)\) relative to the fixed coordinate system.

The condition that there is no slipping at the point of contact \(P\) implies that the velocities of this point on the rotating surface and on the surface of the ball are equal, and is expressed by the non-holonomic constraint
\[ f = r \times \Omega + \dot{r} + R\gamma \times \omega = 0, \quad (1) \]

where \( \gamma = (0, 0, 1) \), \( \omega \) is the angular velocity of the ball referred to the chosen coordinate axes and \( \Omega = (0, 0, \Omega) \) is the angular velocity of the rotating plane on which the ball rolls. Here and below the vectors are denoted in boldface font, the scalar product is denoted by standard round brackets, in which the vectors are written (they are set off by commas), sign ‘\( \times \)’ corresponds to the vector product.

The change in the linear and angular momentum of the ball relative to its centre can be written in the form of Newton–Euler equations:

\[ m\dot{r} = N + F, \quad I\dot{\omega} = RN \times \gamma + M, \quad (2) \]

where \( I \) is the central tensor of inertia of the ball, \( F \) is the resultant of the external active forces, \( M \) is the resulting moment of the external forces relative to the centre of the ball, and \( N \) is the reaction force acting on the ball at the point of contact \( P \) (in the general case its direction can be arbitrary).

Throughout the rest of the paper, we will write all variables and equations in dimensionless form. To do so, we take the radius of the ball \( R \) as the unit of distance, the quantity \( 1/\Omega \) as the unit of time, and the mass of the ball \( m \) as the unit of mass, that is, we make the following change of variables:

\[ \frac{x}{R} \to x, \quad \frac{y}{R} \to y, \quad \frac{t}{\Omega} \to t, \quad \frac{\omega}{\Omega} \to \omega, \quad \frac{g}{mR\Omega^2} \to g. \]

For convenience in analysis of experimental results, the trajectories in figures will be presented in dimensional quantities.

Adding to (2) the derivative of the constraint equation (1) with respect to time, we can eliminate the reaction force and obtain an equation (in dimensionless variables) governing the evolution of the vector \( r \):

\[ (k + 1)\dot{r} - \gamma(\gamma, \dot{r}) + k k \times \gamma = F - \gamma(\gamma, F) + M \times \gamma, \quad (3) \]

where \( k = I/(mR^2) \) with \( k \leq 1: k = 0 \) for a material point and \( k = 2/5 \) for a homogeneous ball.

In the case where the moment of external forces depends on the angular velocity, it is convenient to eliminate the reaction force and \( r, \dot{r} \) from the system of equations (2) taking into account the constraint equation (1). We obtain an equation governing the evolution of the vector \( \omega \):

\[ (k + 1)\dot{\omega} - \gamma(\gamma, \dot{\omega}) = \gamma \times F + M. \quad (4) \]

3. Influence of the inclination of the rotating plane

On the inclined plane \( (\dot{h} > 0) \), if there are no external forces and moments other than the gravity force, \( F = g, \quad M = 0 \).

In this case, equation (3) can be written as

\[ (k + 1)\ddot{\gamma} + k \ddot{\gamma} \times \gamma = g - \gamma(\gamma, g), \]

or, referred to the chosen coordinate axes, as

\[ \dddot{x} + \omega_0^2 \dddot{y} = 0, \quad \dddot{y} - \omega_0^2 k + V_0^2 = 0, \quad (5) \]

where \( V_0^2 = \frac{k \sin \delta}{k} \), \( \omega_0 = \frac{k}{k + 1} \), \( \omega_0 \in [0, 1/2] : \omega_0 = 0 \) for a material point and \( \omega_0 = 2/7 \) for a homogeneous ball.
Integrating \( (5) \), we obtain
\[
\begin{align*}
V_d &= \frac{\dot{x}(0) + V_d}{\omega_0}, \\
\tan \varphi_0 &= -\frac{\dot{x}(0) + V_d}{\dot{y}(0)}.
\end{align*}
\]

Thus, the trajectory of the centre of mass of the ball (and the point of contact) is a trochoid traced out by a point lying inside or outside a circle rolling along a straight line with constant velocity of drift \( V_d \) (figure 2). The constants \( x_c, y_c, C \) and \( \varphi \) are the constants that are defined from the initial conditions:
\[
\begin{align*}
x_c &= x(0) - \frac{\dot{y}(0)}{\omega_0}, \\
y_c &= y(0) + \frac{\dot{x}(0) + V_d}{\omega_0}, \\
C &= \frac{\sqrt{(\dot{x}(0) + V_d)^2 + \dot{y}(0)^2}}{\omega_0}, \\
\tan \varphi_0 &= -\frac{\dot{x}(0) + V_d}{\dot{y}(0)}.
\end{align*}
\]

Thus, it does not depend on initial conditions and is defined only by the angular velocity of the rotating surface on which the ball rolls.

3.1. Analogy with the motion of a charged particle in an electromagnetic field

The motion of a non-relativistic particle with mass \( m \) and charge \( q \) in an electric field with strength \( E \) and in a magnetic field with strength \( H \) is described by the equation [16]
\[
\frac{d\mathbf{r}}{dt} = q\mathbf{E} + \frac{q}{c} \mathbf{v} \times \mathbf{H},
\]

(8)

where \( \mathbf{v} \) is the velocity of the particle and \( c \) is the velocity of light. The left-hand side of (8) is the change in the linear momentum and the right-hand side is the Lorentz force, which, as seen from the expression, is always perpendicular to the velocity of the particle.

In a constant electric field along the axis \( Oy \) and in a magnetic field directed against the axis \( Oz \), the equations of motion (8), referred to the coordinate axes, can be written as

\[
\dot{x} + \frac{qH}{mc} y = 0, \quad \dot{y} - \frac{qH}{mc} x = \frac{qE}{m}.
\]

Denoting \( \omega = \frac{qH}{mc}, V_d = \frac{Ec}{H} \), we obtain precisely equations (5), whose solution has the form (6).

Thus, similarly to the motion of the ball on a rotating plane, the trajectory of a charged particle is a circle whose centre is uniformly displaced along the axis \( Oy \) with drift velocity \( V_d \) depending on the strength of the electric field.

This analogy allows, on the one hand, a qualitative visualisation of the motion of a charged particle in a mechanical laboratory. On the other hand, the effects that are well understood and described in problems of charged particle dynamics can be detected during the motion of a ball on a rotating plane. For example, in [9] an analogy is drawn between the motion of a charged particle in an electromagnetic field with axial symmetry and the motion of a ball on a conical surface.

4. Influence of rolling friction

We further consider the motion of the ball on a rotating plane under the assumption that the ball is acted upon by the moment of rolling friction.

Physically, rolling friction is due to the following factors: deformation of the rolling ball, microimpacts between the ball and the turntable, and adhesion between the moving object and the turntable (the influence of air resistance is insignificant as compared to the above factors). Rolling friction leads to a resultant reaction from the table, which does not pass through the centre of mass of the ball.

In this paper we adopt the hypothesis of viscous rolling friction: the ball is acted upon by the moment of rolling friction, which depends linearly on the velocity, the direction of the corresponding friction torque is opposite to that of the angular velocity of the ball, and the coefficient of proportionality of the rolling friction torque and the angular velocity depends not only on the materials of the contacting surfaces, but also on the mass-geometric characteristics of the ball:

\[
\mathbf{M} = -\alpha (\mathbf{\omega} - \mathbf{\Omega}),
\]

(9)

where \( \alpha \) is the coefficient of proportionality. We consider the simplest case \( \alpha = \text{const} \).

Remark 1. Hypothesis of viscous rolling friction, first, satisfies the condition of rolling without slipping and, second, guarantees dissipation. The adequacy of the rolling friction model under consideration is borne out by theoretical and experimental studies for mechanical systems such as a homogeneous rolling disk [17] and ball [13, 15], a spherical tippe top [18, 19], etc. Taking into account viscous rolling friction (depending on the velocity)
has made it possible to qualitatively describe dynamical phenomena such as retrograde motion of a rolling disk, overturn of a tippe top, etc.

In view of (9) the dimensionless equation (4) can be written as

\[(k + 1)\dot{\omega} - \gamma(\gamma, \dot{\omega}) = \gamma \times F - \bar{\alpha}(\omega - \Omega),\]  

(10)

where \(\bar{\alpha} = \alpha/(mR^2\Omega)\).

It can be seen from (10) that the external force \(F\) does not contribute to the projection of the angular velocity \(\omega\) onto the axis \(Oz\):

\[k\dot{\omega}_z = -\bar{\alpha}(\omega_z - \Omega).\]  

(11)

Thus, the equation for the definition of \(\omega_z\) decouples from the general system of equations of motion and does not influence the motion of the centre of mass of the ball along the trajectory. If the moment of rolling friction has no vertical component, then, according to (11), the angular velocity of the ball about the vertical axis remains unchanged.

**Remark 2.** Since the trajectory of the centre of mass does not depend on the vertical component of the angular velocity, the imposition of an additional non-holonomic constraint \((\omega, \gamma) = 0\), which corresponds to the rubber body model [20], does not lead to a qualitative change in the dynamics of the system (10).

On the inclined plane, if there are no external forces other than the gravity force, substituting (9) into (3) using (1) gives

\[(k + 1)\dot{r} - \gamma(\gamma, \dot{r}) + \bar{\alpha}r + kr \times \gamma + \bar{\alpha}r \times \gamma = g - \gamma(\gamma, g).\]  

(12)

Referred to the chosen coordinate axes, equation (12) can be reduced to a three-parameter linear system of four differential first-order equations relative to the vector of the variables \(z = (x, y, v_x, v_y)\). This system can be represented as

\[\dot{z} = L(z - z_0), \quad L = \begin{pmatrix} 0 & I \\ \eta A & -\eta I + \omega_0 A \end{pmatrix},\]  

(13)

where \(\eta = \bar{\alpha}/(k + 1)\), \(I\) is the unit matrix, \(A_{ij} = -\varepsilon_{ij3}\), and \(z_0\) is the fixed point of the system:

\[z_0 = (-\omega_0 v_y/\eta, 0, 0, 0).\]  

(14)

The eigenvalues of the matrix \(L\) satisfy the equation

\[\lambda^4 + 2\eta\lambda^3 + (\eta^2 + \omega_0^2)\lambda^2 + 2\eta\omega_0\lambda + \eta^2 = 0,\]

are pairwise complex conjugate and have the form

\[\lambda_{1,2} = \frac{\eta + i \omega_0}{2} \pm \frac{\sqrt{\eta^2 - \omega_0^2 + 2i\eta(2 - \omega_0)}}{2},\]  

\[\lambda_{3,4} = \frac{\eta - i \omega_0}{2} \pm \frac{\sqrt{\eta^2 - \omega_0^2 - 2i\eta(2 - \omega_0)}}{2}.\]  

(15)

The roots \(\lambda_k, k = 1 \ldots 4\), depend only on two parameters \(\omega_0\) and \(\eta\). Since for a ball (including a hollow one) rolling on a rotating plane the parameter \(\omega_0\) is known, an investigation of the dependence of the real part of the roots (15) on the parameter \(\eta\) is sufficient for the stability analysis of the fixed point \(z_0\).
For this, we write the real parts of the roots (15) and introduce the following notation:

\[
\alpha_1 = \text{Re}(\lambda_1) = \text{Re}(\lambda_2) = -\frac{\eta}{2} + \frac{\sqrt{2}}{4}\sqrt{(\eta^2 - \omega_0^2)^2 + 4\eta^2(2 - \omega_0)^2 - (\omega_0^2 - \eta^2)},
\]

\[
\alpha_2 = \text{Re}(\lambda_3) = \text{Re}(\lambda_4) = -\frac{\eta}{2} - \frac{\sqrt{2}}{4}\sqrt{(\eta^2 - \omega_0^2)^2 + 4\eta^2(2 - \omega_0)^2 - (\omega_0^2 - \eta^2)}.
\]

(16)

The negativity of \( \alpha_2 \) for any \( \eta > 0 \) follows explicitly from (16).

Next, solving the inequality \( \alpha_1 < 0 \), we obtain for any \( \eta > 0 \) the condition \( \omega_0 > 1 \), which cannot be satisfied by definition of \( \omega_0 \) (see figure 3(a)).

Thus, the following statements are valid:

**Proposition 1.** For any \( \eta > 0 \) and \( \omega_0 \leq 1/2 \) the inequalities

\[ \alpha_1 > 0, \quad \alpha_2 < 0 \]

are satisfied.

**Proposition 2.** The fixed point \( z_0 \) (14) is an unstable focus.

**Remark 3.** The parameter \( V_d \), which depends on the angle of inclination of the plane, does not influence the qualitative behaviour of the system and defines only the shift of the fixed point (14) along the axis \( Ox \) (perpendicular to the direction of the largest slope). The value of \( V_d \) is limited by the angle of inclination of the plane at which slipping begins at the point of contact.

According to (13) and (15), the solution \( z(t) \) is the sum

\[
z(t) = e^{\alpha_1 t}(C_1 e^{i\beta_1 t} + C_2 e^{-i\beta_1 t}) + e^{\alpha_2 t}(C_3 e^{i\beta_2 t} + C_4 e^{-i\beta_2 t}) + z_0,
\]

(17)

where \( C_k, k = 1 \ldots 4 \), are the (complex) constant vectors defined from the initial conditions \( z(0), \beta_1 = \text{Im}(\lambda_1) = -\text{Im}(\lambda_2), \beta_2 = \text{Im}(\lambda_3) = -\text{Im}(\lambda_4) \).

Characteristic time dependences of the first (increasing, \( \sim e^{\alpha_1 t} \)) and the second (decreasing, \( \sim e^{\alpha_2 t} \)) term of the solution (17) are presented in figure 4.

It is seen from figure 4 that the second term decreases to zero without oscillations (it can be shown analytically that the corresponding damping decrement is \( D \gg \omega_0 \)). The first term \( z(t) \) oscillates near zero with an increasing amplitude that corresponds to untwisting spiral
motion of the centre of mass of the ball about some centre whose position is displaced for the whole time of damping of the second term of (17).

The third term in (17) is defined by the inhomogeneity of equation (13) (external force due to non-horizontality of the plane of rolling) and corresponds to displacement of the trajectory by a constant value along the axis Ox (perpendicularly to the direction of the largest slope).

Thus, the trajectory of the centre of mass of the ball is determined by two processes: drift to the fixed point $z_0$ and increase in the radius of the circle tangent to the trajectory. At small $\eta$ at the initial stage of motion, the former process is more essential (a fast displacement occurs with slightly increasing radius, see figure 5(a)). At large $\eta$, by contrast, the increase in the radius of the tangent circle with a slow drift to point $z_0$ is more essential (see figures 5(b), (c)).

As $t \to \infty$, the trajectory of the centre of mass is a spiral with the centre at point $z_0$ for any $\eta$.

At $\delta = 0$ (on a horizontal plane) the third term $z_0$ of (17) is zero. Thus, the trajectory of the centre of mass is a spiral whose centre is displaced to the origin of the coordinate system (with velocity depending on $\eta$). For comparison, figure 6 shows the trajectories of the centre of mass of the ball on two rotating surfaces: a horizontal surface ($\delta = 0$) and an inclined
5. Experimental investigations

The goal of the experimental investigations is to make a qualitative comparison between the experimental trajectories and theoretical trajectories (constructed by taking into account the moment of rolling friction (9)) of the centre of mass of the ball rolling on a horizontal or inclined rotating surface, and to define the value of the coefficient of viscous friction $\alpha$.

For an experimental analysis of the motion dynamics of a homogeneous ball on a rotating plane, we have developed a special laboratory facility—a turntable 250 mm in diameter, made of acrylic plastic. The turntable is caused to rotate by a DC gear motor at angular velocities ranging from $\pi/3$ to $10\pi$ rad s$^{-1}$. The exact value of the angular velocity of rotation of the turntable is recorded by a sensor the angular velocity installed on the output shaft of the motor. The required inclination of the turntable is adjusted by jack legs and is checked using a digital leveling gauge with an accuracy up to $5 \times 10^{-4}$ rad.

The experiments were conducted with three different balls: with a hollow ball with mass $m = 2.7$ g and diameter $d = 40$ mm (ping-pong ball, $k = 2/3$), an aluminium solid ball with mass $m = 12.5$ g and diameter $d = 20$ mm and a rubber-coated steel ball with mass $m = 31.8$ g and diameter $d = 20$ mm.

The ball was set in motion by hand. Before the start of the experiment the ball was held for some time on the turntable by means of a small piece of glossy cardboard until it attained the velocity corresponding to (1) at $\dot{r} = 0$. This was done to avoid a jump in the angular velocity at the initial instant of impact and to ensure the constraint (1).

To film the motion of the ball with a frequency of 30 pictures per second, a high-speed video camera with a resolution of 12 Mp was placed above the turntable. The trajectory of the ball was recovered from video files using a special MATLAB programme by including libraries for working with images.

Further, an averaged value of the initial velocity and the position of the centre of mass at the initial instant of time were recovered from the experimental data. This data was used as initial conditions when numerically integrating the system (13), in which only the parameter $\delta = 0.063$. The heavy solid lines denote circles tangent to the corresponding trajectories at $t = 190$, and the distance between their centres is denoted by $z_0$. 

![Figure 6. Trajectories of the centre of mass of the ball which have been obtained (with the friction torque taken into account) for motion on a plane rotating with angular velocity $\Omega = 2\pi$ rad s$^{-1}$, under the initial conditions $x(0) = 5$ m, $y(0) = 5$ m, $\dot{x}(0) = 0$, $\dot{y}(0) = 1$ m s$^{-1}$, $\eta = 0.001$ at the inclination angle $\delta = 0$ (intermittent line) and $\delta = 0.063$ (solid line).](image-url)
α remained unknown. This coefficient was chosen (on a corresponding scale) in such a way as to reach a (qualitatively) maximal coincidence of the theoretical and experimental curves.

The results obtained for the motion of all three balls at small velocities are in qualitative agreement with the theory, but, as the velocity increased, lighter balls obviously started to slip, resulting in considerable deviations from the calculated trajectories. This is why a rubber-coated ball with the largest available mass was chosen to demonstrate the experimental results (with mass \( m = 31.8 \) g and diameter \( d = 20 \) mm). The rubber coating is the most reliable to ensure that there is no slipping at the point of contact throughout the motion on the turntable.

Examples of characteristic trajectories obtained experimentally for balls moving on a horizontal and an inclined plane versus computed trajectories are presented in figures 7 and 8 (the coordinates \((x, y)\) are expressed in metres and the initial position is denoted by \(z_0\)). A video of the experiments is available at [21].

A series of experiments (no less than ten for the horizontal and the inclined surface) have yielded the value of \( \alpha = (4.8 \pm 0.5) \times 10^{-7} \text{ kg m}^2 \text{ s}^{-1} \) for the pair of materials.
6. Conclusion

In this work we have presented a detailed theoretical and experimental investigation of the problem of a homogeneous ball moving on a rotating plane under the action of constant external force supplemented with the moment of rolling friction which depends linearly on the angular velocity of the ball.

It is interesting that the simple linear model of rolling friction used in this work gives highly satisfactory results explaining the qualitative behaviour of the system. In contrast to studies using more complicated friction models with the pressure distributed over the support on the contact area at a constant friction coefficient or taking into account the Strubeck effect (see [22–27] and references therein), the theoretical results obtained in this work are supported experimentally with a very good accuracy.

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