The Dynamics of Vortex Sources in a Deformation Flow

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Abstract—This paper is concerned with the dynamics of vortex sources in a deformation flow. The case of two vortex sources is shown to be integrable by quadratures. In addition, the relative equilibria (of the reduced system) are examined in detail and it is shown that in this case the trajectory of vortex sources is an ellipse.

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1. INTRODUCTION

In the classical hydrodynamics, the problem of the interaction of n point vortices in an ideal fluid on a plane and a sphere (see, e.g., [7, 11, 19]) is well understood. Its distinctive feature is that the equations of motion of point vortices are represented in Hamiltonian form. Consequently, the dynamics of n point vortices can be explored by applying well-developed methods of Hamiltonian mechanics (the theory of integrability, stability, topological analysis, and perturbation theory).

Along with the above-mentioned model of point vortices, hydrodynamics uses other, more general, vortex models. Historically, the very first model was that of A.A. Fridman and P. Ya. Polubarinova [12] featuring the interaction of more complex point singularities combining vortex properties and the properties of sources and sinks, namely, the model of vortex sources. In some cases, this model is more preferable for the purposes of hydrodynamics (see, e.g., [4]).

It turns out that the equations of motion of vortex sources are not represented in Hamiltonian form (although they possess an invariant measure). In this sense, the system of vortex sources is close to systems arising in nonholonomic mechanics, which result from ideal constraints, but nevertheless, their behavior can be considerably different from Hamiltonian systems [5, 9].

The system of two vortex sources is integrated in [8], and the integrability of three arbitrary vortex sources is shown in [2]. We note that a reduced system on the so-called form sphere has been obtained using reduction in [2]. This system describes the evolution of configurations of vortex sources up to similarity. Possible phase portraits and various relative equilibria of the system are presented.

Among other, more general, models we mention the model of mass vortices [10] (i.e., vortices that possess not only vorticity, but also mass) and diverse variations, considered in theoretical physics, of dipole approximation [18, 21]. Unfortunately, in spite of such a variety of models, the possibility of their practical application remains unclear so far.

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The motion of point and distributed vortices in a background (deformation) flow was discussed in several publications [3, 13, 15, 22–24]. In this paper, we consider vortex sources in a deformation flow. Their special feature is that they have no Poisson structure. We examine in detail the case of two vortex sources and the method of their integration and qualitative analysis.

2. EQUATIONS OF MOTION

Consider the planar flow of an ideal incompressible fluid with a velocity field which in a Cartesian coordinate system $Oxy$ can be represented as $\mathbf{v} = (v_x, v_y)$.

We express the components of velocity $\mathbf{v}$ in terms of the derivatives of the stream function $\psi$:

$$v_x = \frac{\partial \psi}{\partial y}, \quad v_y = -\frac{\partial \psi}{\partial x}.$$  \hspace{1cm} (2.1)

Then, rewriting the Euler equations using (2.1), we obtain an equation determining the stream function (for more details, see [14]):

$$\frac{\partial \Delta \psi}{\partial t} + \frac{\partial \psi}{\partial y} \frac{\partial}{\partial x} \Delta \psi - \frac{\partial \psi}{\partial x} \frac{\partial}{\partial y} \Delta \psi = 0,$$ \hspace{1cm} (2.2)

where $\Delta$ is the Laplace operator.

As is well known, for the stationary stream function $\psi = \psi(x, y)$ Eq. (2.2) can be represented as

$$\Delta \psi = F(\psi),$$

where $F(\psi)$ is an arbitrary function.

Consider a (partial) stationary solution of (2.2) describing the motion of $n$ vortex sources in a deformation flow, which can be represented as

$$\psi = \psi_v + \psi_d.$$

The function $\psi_v$ is the stream function of $n$ (point) vortex sources

$$\psi_v = -\frac{1}{4\pi} \sum_{i=1}^{n} \Gamma_i \ln((x-x_i)^2 + (y-y_i)^2) + \frac{1}{2\pi} \sum_{i=1}^{n} K_i \arctg \left( \frac{y-y_i}{x-x_i} \right),$$

where $(x_i, y_i)$ are the Cartesian coordinates of the $i$-th vortex source, and $\Gamma_i$ and $K_i$ are the vorticity and intensity of the source, which we assume to be constant (see [12]).

The function $\psi_d$ defines the deformation flow of the fluid with its center at the origin of coordinates

$$\psi_d = \frac{1}{2} ax^2 + \frac{1}{2} by^2.$$ \hspace{1cm} (2.3)

In what follows we assume that the coordinate system $Oxy$ has been chosen in such a way that $a > b$ holds. In this case, three qualitatively different cases are possible:

— if $b = 0$, the flow of the fluid is a shear (plane-parallel) flow with a linear velocity field [3];
— if $b > 0$, the stream lines (2.3) are a family of ellipses with constant eccentricity (see Fig. 1a);
— if $b < 0$, the stream lines (2.3) are a family of hyperbolas (see Fig. 1b).

Remark. Among other generalizations of the problem of the motion of point vortices we mention those presented in [1, 23]. In [1], the motion of a vortex pair is considered in the case where a vortex source has been placed at the origin of coordinates, and in [23] a fixed vortex is placed at the origin of coordinates. The motion of the vortex pair (and of the vortex sources) in the field of one fixed vortex source remains integrable, which follows from [2].
Remark. We present other particular solutions of the system (2.2) describing the (potential) stream function of a vortex array with period $L$ and with vortex intensities $\Gamma$ (see [20]):

$$\psi_d = -\frac{\Gamma}{4\pi} \ln \left( \frac{1}{2} \text{ch} \frac{2\pi y}{L} - \frac{1}{2} \cos \frac{2\pi x}{L} \right),$$

and the (nonpotential) stream function describing the Stuart vortices [25]:

$$\psi_d = \ln(k \text{ch} y + \sqrt{k^2 - 1} \cos x), \quad k \in (1, \infty).$$

The equations of motion of $n$ vortex sources in a deformation flow have the form

$$\begin{align*}
\dot{x}_i &= -\frac{1}{2\pi} \sum_{j \neq i}^n \frac{\Gamma_j (y_i - y_j) - K_j (x_i - x_j)}{(x_i - x_j)^2 + (y_i - y_j)^2} + by_i, \\
\dot{y}_i &= \frac{1}{2\pi} \sum_{j \neq i}^n \frac{\Gamma_j (x_i - x_j) + K_j (y_i - y_j)}{(x_i - x_j)^2 + (y_i - y_j)^2} - ax_i,
\end{align*} \quad (2.4)$$

where $i = 1, \ldots, n$.

Remark. If $a = b$, then the deformation flow is a rotation of the fluid with constant angular velocity. In this case, after transition to a rotating coordinate system the problem reduces to investigating $n$ vortex sources in the absence of a deformation flow.

The system (2.4) preserves the standard invariant measure

$$\mu = \prod_{i=1}^n dx_i dy_i.$$

However, in the general case it is not Hamiltonian. Let us define the vector fields $u_x$ and $u_y$ corresponding to the shifts along the axes $Ox$ and $Oy$

$$u_x = \sum_{i=1}^n \frac{\partial}{\partial x_i}, \quad u_y = \sum_{i=1}^n \frac{\partial}{\partial y_i}, \quad (2.5)$$

and denote the vector field of the system (2.4) by $u$. These vector fields form a solvable Lie algebra with respect to commutation operations:

$$[u_x, u] = -au_y, \quad [u_y, u] = bu_x, \quad [u_x, u_y] = 0. \quad (2.6)$$

Hence, according to the Lie theorem (see, e.g., [16]), one can reduce the order of the system (2.4) by two by choosing the integrals (2.5) as new variables. For the case of two vortex sources in a deformation flow the system (2.4) is integrable by the Euler–Jacobi–Lie theorem [17].
Remark. Let us find out to what type the algebra of the vector fields (2.6) belongs. Since \( a > b \), several cases are possible. If \( b = 0 \), then the vector fields form the Heisenberg algebra:

\[
e_1 = u_y, \quad e_2 = \frac{u_x}{\sqrt{a}}, \quad e_3 = \frac{u}{\sqrt{a}},
\]

\[
[e_1, e_2] = 0, \quad [e_1, e_3] = 0, \quad [e_2, e_3] = e_1.
\]

In the case \( b \neq 0 \) we define new generatrices of the group (2.6)

\[
e_1 = \frac{u_y}{\sqrt{a}}, \quad e_2 = \frac{u_x}{\sqrt{|b|}}, \quad e_3 = \frac{u}{\sqrt{a|b|}},
\]

if \( b > 0 \), we obtain the algebra \( e(2) \):

\[
[e_1, e_2] = 0, \quad [e_1, e_3] = e_2, \quad [e_2, e_3] = e_1,
\]

while if \( b < 0 \), we obtain the algebra \( e(1,1) \):

\[
[e_1, e_2] = 0, \quad [e_1, e_3] = e_2, \quad [e_3, e_2] = -e_1.
\]

3. TWO VORTEX SOURCES IN A DEFORMATION FLOW

In this section, we consider the case \( n = 2 \) and perform an explicit reduction of order. To do so, we pass to new variables in the system (2.4):

\[
z_1 = x_2 - x_1, \quad z_2 = y_1 - y_2,
\]

\[
u = \frac{1}{2} \sum_{i=1}^{2} (a \Gamma_i x_i + b K_i y_i), \quad v = a \sum_{i=1}^{2} (\Gamma_i y_i - K_i x_i).
\]

Since \( u_x(z_i) = u_y(z_i) = 0, \ i = 1, 2 \), the equations of motion for \((z_1, z_2)\) form the following reduced system of equations:

\[
\dot{z}_1 = -b z_2 + \frac{c_1 z_2 - c_2 z_1}{z_1^2 + z_2^2}, \quad \dot{z}_2 = a z_1 - \frac{c_1 z_1 + c_2 z_2}{z_1^2 + z_2^2} = -\frac{\partial H}{\partial z_1},
\]

\[
H = \frac{c_1}{2} \ln(z_1^2 + z_2^2) - c_2 \arctg \frac{z_2}{z_1} - \frac{1}{2} a z_1^2 - \frac{1}{2} b z_2^2,
\]

\[
c_1 = \frac{\Gamma_1 + \Gamma_2}{2\pi}, \quad c_2 = \frac{K_1 + K_2}{2\pi},
\]

where in the general case (when \( c_2 \neq 0 \)) \( H \) is not a single-valued function. Thus, the problem reduces to investigating the vector field (3.2) defined on the two-dimensional manifold, which is the plane \( \mathbb{R}^2 \) with the punctured point \( z_1 = z_2 = 0 \).

We note that the system (3.2) preserves a standard invariant measure and hence, according to the Euler–Jacobi theorem, its solution is represented by quadratures. They are rather cumbersome and therefore inapplicable to the study of dynamics.

In order to define the motion of the vortices using the known solutions \( z_1(t) \) and \( z_2(t) \), it is necessary to integrate the equations of motion for \((u, v)\), which have the form

\[
\dot{u} = -bv - \frac{(a - b)c_3 z_1}{z_1^2 + z_2^2}, \quad \dot{v} = au,
\]

\[
c_3 = \frac{\Gamma_1 K_2 - \Gamma_2 K_1}{2\pi}.
\]

We consider in more detail the reduced system (3.2) in the case where the level lines of the deformation flow are limited (i.e., \( a > b > 0 \)) and \( c_1 > 0 \). We note that in this case the system (3.2) possesses fixed points which characterize its phase portrait.
3.1. The Dynamics of a Reduced System

The system (3.2) has singularity at the point \( z_1 = z_2 = 0 \), at which two vortex sources merge. In order to explore the dynamics of the system (3.2) in its neighborhood, we rescale time as \( dt = (z_1^2 + z_2^2) d\tau \). As a result, the equations of motion take the form

\[
\frac{dz_1}{d\tau} = -b(z_1^2 + z_2^2)z_2 + c_1 z_2 - c_2 z_1, \quad \frac{dz_2}{d\tau} = a(z_1^2 + z_2^2)z_1 - c_1 z_1 + c_2 z_2.
\]  

The eigenvalues of the linearization matrix of the system (3.4) in a neighborhood of \( z_1 = z_2 = 0 \) are

\[ \lambda_{1,2} = -c_2 \pm ic_1. \]

Hence, the following proposition holds.

**Proposition 1.** The fixed point \( z_1 = z_2 = 0 \) of the system (3.4) with \( c_2 \neq 0 \) and \( c_1 \neq 0 \) is a focus that is stable for \( c_2 > 0 \) and unstable for \( c_2 < 0 \).

**Case** \( c_2 = 0 \). If the sum of source intensities is equal to zero, then the system (3.2) possesses the following fixed points (which are singular points of the function \( H \)):

1) isolated equilibrium points

\[ z_1 = 0, \quad z_2 = \pm \sqrt{\frac{c_1}{b}}, \]

in whose neighborhood the eigenvalues of the linearization matrix have the form

\[ \lambda_{1,2} = \pm i \sqrt{2b(a-b)}, \]

hence the above-mentioned singular point is of center type;

2) isolated equilibrium points

\[ z_1 = \pm \sqrt{\frac{c_1}{a}}, \quad z_2 = 0, \]

in whose neighborhood the eigenvalues of the linearization matrix have the form

\[ \lambda_{1,2} = \pm \sqrt{2a(a-b)}, \]

thus, the above-mentioned singular point is of saddle type.

A typical phase portrait of the system (3.2) with \( c_2 = 0 \) is presented in Fig. 2a.

**Case** \( c_2 \neq 0 \). In this case, we parameterize the fixed points as follows:

\[ z_1 = s, \quad z_2 = \frac{1}{s} \frac{c_2}{a-b}. \]

Then it follows from (3.4) that \( s \) is the root of the biquadratic equation:

\[ as^4 - c_1 s^2 + \frac{b c_2^2}{a-b} = 0. \]

Thus, in the case of a small deviation of \( c_2 \) from zero, the equilibrium points are preserved (see Fig. 2b), but as the critical value \( c_2 = \pm c_2^* \) is attained, they disappear:

\[ c_2^* = \frac{c_1}{2} \frac{a-b}{\sqrt{ab}}. \]

Then the system (3.2), except for \( z_1 = z_2 = 0 \), has no equilibrium points. It is also easy to show that in this case too there are no particular periodic solutions (see Fig. 2c). Hence, in this case the following proposition holds.
Proposition 2. If \( c_2 \notin (-c_2^*, c_2^*) \), then two vortex sources either merge for \( c_2 > 0 \) or move infinitely away from each other for \( c_2 < 0 \), (i.e., we have either a collapse or a recession).

Consider a deformation flow with coefficients periodically depending on time (see [15] for details):

\[
\begin{align*}
    a &= a_0(1 - \epsilon b_0 \sin(\nu t)), \\
    b &= b_0(1 + \epsilon a_0 \sin(\nu t)).
\end{align*}
\]

If we introduce a new variable \( \omega = \nu t \), then after rescaling time as \( dt = (z_1^2 + z_2^2) d\tau \) we can represent the system (3.4) as

\[
\begin{align*}
    \frac{dz_1}{d\tau} &= -b_0(1 + \epsilon a_0 \sin \omega)z_2(z_1^2 + z_2^2) + c_1 z_2 - c_2 z_1, \\
    \frac{dz_2}{d\tau} &= a_0(1 - \epsilon b_0 \sin \omega)z_1(z_1^2 + z_2^2) - c_1 z_1 - c_2 z_2, \\
    \frac{d\omega}{d\tau} &= (z_1^2 + z_2^2) \nu.
\end{align*}
\]

In the case \( c_2 = 0 \) the system (3.5) is analogous to a system with one and a half degrees of freedom. The Poincaré map is presented in Fig. 3, which shows the presence of chaotic trajectories and nonintegrability of the system.

Fig. 2. Phase portrait of the system (3.2) depending on the parameter \( c_2 \) for fixed \( a = 4, b = 2, c_1 = 3 \).

Fig. 3. The poincaré map of the system (3.5) for fixed parameters \( a = 4, b = 2, c_1 = 3, c_2 = 0, \nu = 1 \) and the secant plane \( \omega = 0 \).
Obviously, the result that two vortex sources with \( c_2 \neq 0 \) either merge (when \( c_2 > 0 \)) or infinitely move from each other (when \( c_2 < 0 \)) remains valid.

3.2. Absolute Dynamics

For two vortex sources the following proposition holds.

**Proposition 3.** The motion of vortex sources along the known solutions \( z_1(t) \) and \( z_2(t) \) from the system (3.2) is restored by means of quadratures.

**Proof.** Indeed, in this case \( u(t) \) and \( v(t) \) satisfy the linear system

\[
\dot{u} = -bv - f(t), \quad \dot{v} = au, \quad f(t) = \frac{(a - b)c_3z_1(t)}{z_1(t)^2 + z_2(t)^2}.
\]

(3.6)

We represent the solution of the system (3.6) as

\[
\begin{align*}
   u(t) &= \cos(\omega t) \left( u_0 - \xi_2(t) \right) - \sin(\omega t) \left( v_0 \frac{\sqrt{b}}{\sqrt{a}} + \xi_1(t) \right), \\
   v(t) &= \cos(\omega t) \left( v_0 + \frac{\sqrt{a}}{\sqrt{b}} \xi_1(t) \right) + \frac{\sqrt{a}}{\sqrt{b}} \sin(\omega t) \left( u_0 - \xi_2(t) \right), \\
   \xi_1(t) &= \int_0^t f(\tau) \sin(\omega \tau) d\tau, \quad \xi_2(t) = \int_0^t f(\tau) \cos(\omega \tau) d\tau,
\end{align*}
\]

(3.7)

where \( \omega = \sqrt{ab} \), and \( u_0 \) and \( v_0 \) are the initial values of \( u \) and \( v \), respectively.

Using (3.7) from (3.1), we find the quadratures for the coordinates of both vortex sources. \( \square \)

Analysis of the quadratures (3.7) is a nontrivial problem (for a qualitative analysis of the point of contact for the Chaplygin ball, see, e.g., [6]). We apply them for analysis of the motion of vortex sources at equilibrium points of the system (3.2).

Let \((\tilde{z}_1, \tilde{z}_2)\) be the fixed points of the system (3.2) which were considered in the previous section. In this case, \( f(t) = f_0 = \text{const.} \) Then from (3.7) we find the trajectory of the first vortex source:

\[
\begin{align*}
   x_1^{(0)} &= \frac{\tilde{z}_1 \Gamma_2 (\Gamma_1 + \Gamma_2) a^2 + ab (K_2^2 \tilde{z}_1 + (\Gamma_1 \tilde{z}_2 + K_1 \tilde{z}_1 K_2 - K_1 \Gamma_2 \tilde{z}_2) - f_0 (K_1 + K_2)}{a (a (\Gamma_1 + \Gamma_2)^2 + b (K_1 + K_2)^2)}, \\
   y_1^{(0)} &= \frac{\tilde{z}_2 \Gamma_2 (K_1 + K_2) b^2 + ab (\Gamma_2^2 \tilde{z}_2 + (\Gamma_1 \tilde{z}_2 + K_1 \tilde{z}_1 K_2 - K_2 \Gamma_1 \tilde{z}_1) - f_0 (\Gamma_1 + \Gamma_2)}{b (a (\Gamma_1 + \Gamma_2)^2 + b (K_1 + K_2)^2)}, \\
   R_1 &= \frac{abu_0^2 + (bv_0 + f_0)^2}{a (a (\Gamma_1 + \Gamma_2)^2 + b (K_1 + K_2)^2)}.
\end{align*}
\]

In order to obtain the trajectory of the second vortex source, we have to make a corresponding change of indices \((1 \to 2 \text{ and } 2 \to 1)\) in the previous relations. Thus, in this case each vortex source rotates in an ellipse with the center displaced relative to the origin of coordinates (see Fig. 4). It is easy to show that both vortex sources rotate with the same angular velocity equal to \( \sqrt{ab} \).

4. VORTICES IN A DEFORMATION FLOW

In the case of vortices in a deformation flow (i.e., \( K_i = 0, \ i = 1, \ldots, n \)) the equations of motion (2.4) possess the integral \( H_v \) and are represented in the Hamiltonian form

\[
\begin{align*}
   \dot{x}_i &= \frac{1}{\Gamma_i} \frac{\partial H_v}{\partial y_i}, \quad \dot{y}_i = -\frac{1}{\Gamma_i} \frac{\partial H_v}{\partial x_i}, \quad i = 1, \ldots, n, \\
   H_v &= -\frac{1}{4\pi} \sum_{i \neq j}^n \Gamma_i \Gamma_j \ln((x_i - x_j)^2 + (y_i - y_j)^2) + \frac{a}{2} \sum_{i=1}^n \Gamma_i x_i^2 + \frac{b}{2} \sum_{i=1}^n \Gamma_i y_i^2.
\end{align*}
\]

(4.1)
Fig. 4. Trajectories of vortex sources for fixed points of the system (3.2) and the fixed parameters $a = 4$, $b = 2$, $u_0 = 14$, $v_0 = 17$.

The Poisson bracket corresponding to the system (4.1) has the form

$$\{f, g\} = \sum_{i=1}^{n} \frac{1}{\Gamma_i} \left( \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial x_i} \right).$$

With the help of this Poisson bracket the vector fields $u_x$ and $u_y$ are also represented in Hamiltonian form if one chooses, respectively,

$$P_v = \sum_{i=1}^{n} \Gamma_i x_i, \quad Q_v = \sum_{i=1}^{n} \Gamma_i y_i$$

as the Hamiltonian.

In this case, $P_v$ and $Q_v$ are not in involution with the Hamiltonian $H_v$:

$$\{P_v, H_v\} = bQ_v, \quad \{Q_v, H_v\} = -aP_v, \quad \{P_v, Q_v\} = \sum_{i=1}^{n} \Gamma_i. \quad (4.2)$$

The commutator relations (4.2) do not form a Lie–Poisson structure, since they contain the cocycle $\left( \sum_{i=1}^{n} \Gamma_i \right)$.

The Casimir function (4.2) and hence the additional integral (4.1) have the form

$$F_v = aP_v^2 + bQ_v^2.$$ 

Thus, the following assertions hold:

— The system (4.1) admits order reduction by one degree of freedom. As a consequence, the three-vortex problem reduces to a (Hamiltonian) system with two degrees of freedom;

— The relations $P_v = 0$ and $Q_v = 0$ define the invariant manifold of the system (4.1). In this case, when $\sum_{i=1}^{n} \Gamma_i = 0$, the system (4.1) admits order reduction by two degrees of freedom. The motion of three vortices in this case is integrable.
5. THE CASE OF PROPORTIONAL SOURCE INTENSITIES AND VORTICITIES

The above results are generalized in the case of proportional source intensities and vorticities:

\[ \Gamma_i = \mu K_i, \quad \mu = \text{const}, \quad i = 1, \ldots, n. \]  

(5.1)

In this case, the equations of motion (2.4) have the form

\[ \dot{x}_i = \frac{1}{K_i} \frac{\partial H_s}{\partial y_i}, \quad \dot{y}_i = -\frac{1}{K_i} \frac{\partial H_s}{\partial x_i}, \quad i = 1, \ldots, n \]

\[ H_s = -\frac{\mu}{4\pi} \sum_{i \neq j} K_i K_j \ln((x_i - x_j)^2 + (y_i - y_j)^2) + \frac{1}{2\pi} \sum_{i \neq j} K_i K_j \arctg \frac{y_j - y_i}{x_i - x_j} + \]

\[ + \frac{a}{2} \sum_{i=1}^{n} K_i x_i^2 + \frac{b}{2} \sum_{i=1}^{n} K_i y_i^2, \]

where \( H_s \) is generally a many-valued function. The system (5.2) possesses the additional integral \( F_s \):

\[ F_s = a P_s^2 + b Q_s^2, \quad P_s = \sum_{i=1}^{n} K_i x_i, \quad Q_s = \sum_{i=1}^{n} K_i y_i. \]  

(5.3)

In this case, the integrability of the system of two vortex sources follows from the existence of the integral (5.3) and the Liouville theorem. Unfortunately, the geometrical version of foliation by tori (the Liouville – Arnold theorem) cannot be applied here due to the many-valuedness of \( H_s \) and noncompactness of the trajectories.

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The computer simulation was carried out using the software package “Computer Dynamics: Chaos” (http://lab-en.ics.org.ru/lab/page/kompyuternaya-dinamika/).

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