The Vlasov Kinetic Equation, Dynamics of Continuum and Turbulence

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Abstract—We consider a continuum of interacting particles whose evolution is governed by the Vlasov kinetic equation. An infinite sequence of equations of motion for this medium (in the Eulerian description) is derived and its general properties are explored. An important example is a collisionless gas, which exhibits irreversible behavior. Though individual particles interact via a potential, the dynamics of the continuum bears dissipative features. Applicability of the Vlasov equations to the modeling of small-scale turbulence is discussed.

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1. INTRODUCTION

A continuous medium is a continuum of interacting particles. Therefore, the kinetic equations should play a key role in describing the evolution.

It is typical to use the Boltzmann kinetic equation (historically it is the first) or its modifications arising in the analysis of the Bogoliubov chain of equations (the B-B-G-K-Y hierarchy) (see [1–3]). Some authors regard Bogoliubov’s approach as too formal (see [4] for a discussion of this). However, this approach makes it possible to obtain hydrodynamic equations taking viscosity into account.

There is a fundamental difficulty here, though, which has not been overcome until now. The point is that the original Newton equations (at the microlevel) are reversible: they are invariant under a simultaneous reversal of time and velocity.

\[ t \mapsto -t, \quad v \mapsto -v. \]  

However, the Boltzmann kinetic equation (and its modifications) are irreversible as are the Navier–Stokes equations for viscous flows. Therefore, it does not seem possible to give a rigorous justification of the Bogoliubov method.

This paper develops an approach based on the use of the Vlasov kinetic equation:

\[ \frac{\partial \rho^*}{\partial t} + \frac{\partial \rho^*}{\partial x_i} u_i - \frac{\partial \rho^*}{\partial u_j} \frac{\partial}{\partial x_j} \int K(x, y) \rho^*(y, w, t) \, dw \, dy = 0. \]  

The equation governs the evolution of a continuum of particles with the pair interaction potential K. We shall assume that \( K(x, y) = K(y, x) \). It is generally believed that the kernel K depends only on the distance \(|x - y|\). Finding the interaction potential is only possible within the framework of quantum mechanics. Of course, equation (1.2) admits involution (1.1).

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602
In equation (1.2) \( \{x_i\}_n \) are the coordinates, and \( \{u_i\}_n \) are the velocities of interacting particles. The dimension \( n \) of the Euclidean configuration space is irrelevant. The density \( \rho^* \) of the medium is the function of \( x, u \) and \( t \). It is non-negative, and 
\[
\int \int \rho^*(x, u, t) \, dx \, du = m
\]
is the mass of matter. The summation is extended over the recurring indices.

Equation (1.2) does not take external forces into account. In fact, it can be done quite simply: to the kernel \( K \) one should add a function that does not depend on the variables \( y \). The choice of the pair interaction potential \( K \) depends on the problem under consideration. For example, if the equilibrium configurations of a rotating self-gravitating fluid are investigated, \( K \) will define gravitation, while for solid gases it is natural to use the semi-empirical Lennard-Jones potential as function \( K \). By the way, in the absence of interaction (when \( K = 0 \)) we obtain the Poincaré–Knudsen conceptual model of collisionless gas. The Vlasov equation can also be written for other types of interactions (e.g., the electromagnetic interaction [5, 6]).

The Vlasov equation possesses the following remarkable property. For any integer \( N \) it admits a generalized solution of the form
\[
\rho^*(x, u, t) = \sum_{i=1}^{N} m_i \delta(x - x_i(t)) \delta(u - u_i(t)).
\] (1.3)

Singularities are interpreted as states of the system of \( N \) particles and the coefficients as their masses. If (1.3) is a generalized solution of (1.2), the coordinates and velocities of these \( N \) particles satisfy ordinary Newton’s differential equations that govern the evolution of the system of \( N \) interacting particles with potential \( K \). Recall that the density of any distribution can be approximated (in the weak sense) as accurately as desired by weighted sums (1.3). This idea underlies the proofs of the theorems on the existence of both generalized and classical solutions of the Vlasov equation (see, e.g., [7–9]).

Since the Vlasov equation is reversible, the fundamental question remains: are the solutions of this equation capable of describing dissipative processes? In other words, can the continuum of interacting particles exhibit irreversible macroscopic behavior? The answer turns out to be positive. The solution to this problem is based on the analysis of the properties of weak limits of solutions to equation (1.2) as time grows infinitely. This approach is developed in [10].

The goal of this paper is to derive the Euler equations of motion for continuous media, which are governed by the Vlasov equation, and to explore their general properties. In some essential features they differ from the classical equations of aero- and hydrodynamics. Even in the simplest case where \( K = 0 \), these equations do not coincide in their form and properties with the classical equations for an ideal gas.

2. EQUATIONS OF MOTION

We make use of the Euler approach and present the basic characteristics of a continuous medium as functions of coordinates \( x \) and time \( t \). They are given by averaging dynamical quantities over velocities (see [3] and [5] for more on this). We introduce the density of the medium in configuration space
\[
\rho(x, t) = \int \rho^*(x, u, t) \, du
\] (2.1)
and the velocity field
\[
v_i(x, t) = \left[ \frac{u_i \rho^*(x, u, t) \, du}{\int \rho^*(x, u, t) \, du} \right] = \frac{\int u_i \rho^* \, du}{\rho}.
\] (2.2)

We first derive the equation of continuity which expresses the law of conservation of mass:
\[
\frac{\partial \rho}{\partial t} = \int \frac{\partial \rho^*}{\partial t} \, du = - \int \frac{\partial \rho^*}{\partial x_i} u_i \, du + \int \frac{\partial \rho^*}{\partial u_i} \frac{\partial V}{\partial x_i} \, du,
\] (2.3)
where
\[
V(x, t) = \int K(x, y) \rho^*(y, w, t) \, dw \, dy = \int K(x, y) \rho(y, t) \, dy. \tag{2.4}
\]
In view of (2.2) the first term on the right-hand side of (2.3) takes the form
\[- \frac{\partial}{\partial x_i} \int u_i \rho^* \, du = - \frac{\partial}{\partial x_i} (\rho v_i). \]
And the second term on the right-hand side of (2.3) becomes zero since
\[
\int \frac{\partial \rho^*}{\partial u_i} \frac{\partial V}{\partial x_i} \, du = \frac{\partial V}{\partial x_i} \int \frac{\partial \rho^*}{\partial u_i} \, du = 0,
\]
if \( \rho^* \to 0 \) as \( |u| \to \infty \). In what follows we shall retain this natural assumption.

Thus, relation (2.3) becomes the classical equation of continuity
\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j} (\rho v_j) = 0. \tag{2.5}
\]
As is shown in [5], this equation also holds for more general assumptions of the kinetics of a medium.

We now derive the Euler equation, which expresses the momentum theorem for the system:
\[
\frac{\partial \rho v_i}{\partial t} = \int u_i \frac{\partial \rho^*}{\partial t} \, du = - \int u_i \frac{\partial \rho^*}{\partial x_j} u_j + \int u_i \frac{\partial \rho^*}{\partial u_j} \frac{\partial V}{\partial x_j} \, du = \frac{\partial}{\partial x_j} \int u_i u_j \rho^* \, du + \int u_i \frac{\partial \rho^*}{\partial u_j} \frac{\partial V}{\partial x_j} \, du, \tag{2.6}
\]
Let us transform the integral in the last term:
\[
\int u_i \frac{\partial \rho^*}{\partial u_j} \, du = - \int \rho^* \delta_{ij} \, du = - \rho \delta_{ij}. \tag{2.7}
\]
After elementary transformations equation (2.6) (in view of (2.7)) takes the form of the Euler equation:
\[
\frac{\partial \rho v_i}{\partial t} + \frac{\partial}{\partial x_j} (\rho v_i v_j) = - \frac{\partial P_{ij}}{\partial x_j} - \rho \frac{\partial V}{\partial x_i}, \tag{2.8}
\]
where
\[
P_{ij}(x, t) = \int (u_i - v_i)(u_j - v_j) \rho^*(x, u, t) \, du \tag{2.9}
\]
is a stress tensor. Obviously, it is symmetric: \( P_{ij} = P_{ji} \).

The function \( V \) can be interpreted as density of the potential of “internal” forces. The last term in (2.8) has the meaning of a force with which the medium acts on the particle at point \( x \) at time \( t \). It is present due to long-range action being taken into account: distant particles also have an effect on the motion of the medium. The phenomenological approach does not usually take it into account.

In the general case the stress tensor \( P_{ij} \) is not spherical. It will be spherical only under some additional conditions: for example, when the density \( \rho^* \) depends on the squared velocity. The latter condition is obviously satisfied if \( n = 1 \). In this case the integral (2.9)
\[
p = \int (u - v)^2 \rho^* \, du
\]
will be the pressure of the medium in a flow. In phenomenological models one usually sets \( P_{ij} = p \delta_{ij} \) for \( n > 1 \).
According to (2.4), the Euler equation (2.8) is differentially integral. If desired, the equation governing the evolution of potential V can be added to it. To do so, we differentiate relation (2.4) with respect to \( t \) and use the equation of continuity (2.5):

\[
\frac{\partial V}{\partial t} = -\int K(x, y) \frac{\partial}{\partial y_i} (\rho v_i) \, dy.
\]  

(2.10)

This equation will also be integral since the integrand contains unknown functions \( \rho \) and \( v_1, \ldots, v_n \). Equation (2.10) can be slightly simplified if one assumes that the potential \( K \) depends only on the difference \( |x - y| \), has no singularities and

\[
K(|x - y|) \rho(y, t) v_i(y, t) \to 0
\]  

(2.11)

as \( |y| \to \infty \). Indeed, integration by parts gives:

\[
\frac{\partial V}{\partial t} = -\frac{\partial}{\partial x_i} \int K(|x - y|) \rho v_i \, dy.
\]  

(2.12)

Here, we have used another simple relation:

\[
\frac{\partial}{\partial x_i} K(|x - y|) = -\frac{\partial}{\partial y_i} K(|x - y|).
\]  

(2.13)

Note that instead of condition (2.11) it suffices to assume that we consider a problem with periodic boundary conditions: all functions (including the potential \( K \)) are periodic in \( x_1, \ldots, x_n \).

It is clear that the system of equations (2.5) and (2.8) is not closed: it must be supplemented with a relation for the stress tensor. Proceeding along these lines, we obtain additional equations containing new velocity moments and so on. As a result, the exact equations of motion will constitute an infinite chain of coupled equations. This circumstance has already been pointed out by A. A. Vlasov [5]. When the phenomenological approach is used, one usually terminates the chain of equations at the third step, adding algebraic relations between the basic variables. However, these “equations of state” only hold under some additional assumptions (like the conditions of local equilibrium or local reversibility). Moreover, the solutions of such “truncated” equations possess entirely different qualitative properties.

3. THE INFINITE CHAIN OF EQUATIONS \((n = 1)\)

The above-mentioned chain of coupled equations has the simplest form in the one-dimensional case. We shall assume that the density \( \rho^* \) very quickly decreases at infinity in velocity:

\[
u^k \rho^*(x, u, t) \to 0
\]

as \( |u| \to \infty \) for all \( k \geq 0 \).

Set

\[
f_0(x, t) = \int_{-\infty}^{\infty} \rho^*(x, u, t) \, du = \rho,
\]

\[
f_k(x, t) = \int_{-\infty}^{\infty} u^k \rho^*(x, u, t) \, du \bigg/ \int_{-\infty}^{\infty} \rho^*(x, u, t) \, du, \quad k \geq 1.
\]  

(3.1)

We have (for \( k \geq 1 \)):

\[
\frac{\partial \rho f_k}{\partial t} = \int u^k \frac{\partial \rho^*}{\partial t} \, du = -\int u^{k+1} \frac{\partial \rho^*}{\partial x} \, du + \frac{\partial V}{\partial x} \int u^k \frac{\partial \rho^*}{\partial u} \, du
\]

\[
= -\frac{\partial}{\partial x} (\rho f_{k+1}) - \frac{\partial V}{\partial x} k \rho f_{k-1}.
\]

It is simpler to express these equations in terms of new variables \( \{g_k\}_0^\infty \):

\[
g_0 = f_0, \quad g_k = \rho f_k \quad (k \geq 1).
\]
As a result, we obtain an infinite system of equations of motion:

\[
\frac{\partial g_k}{\partial t} + \frac{\partial g_{k+1}}{\partial x} + kg_{k-1} \frac{\partial V}{\partial x} = 0, \quad k = 0, 1, 2, \ldots,
\]

\[
\frac{\partial V}{\partial t} + \int K(x, y) \frac{\partial g_1}{\partial y} \, dy = 0.
\]

(3.2)

For \( k = 0 \) we formally obtain an equation of continuity (the meaning of the symbol \( g_{-1} \) is inessential). If \( K \) depends on the difference \(|x - y|\), then in the absence of singularities the second equation (which coincides with equation (2.10)) can be rewritten in gradient form:

\[
\frac{\partial V}{\partial t} + \frac{\partial}{\partial x} \int K(|x - y|) g_1 \, dy = 0.
\]

(3.3)

If \( K \neq 0 \), equations (3.2) are nonlinear. In fact, the second differential equation (3.2) (or equation (3.3)) can be replaced with the “finite” functional relation

\[
V = \int K(x, y) g_0(y, t) \, dy.
\]

(3.4)

**Example.** Let \( K = \delta(x - y) \) (elastic repulsion). Then \( V = \rho \), and equations (3.2) become:

\[
\frac{\partial g_k}{\partial t} + \frac{\partial g_{k+1}}{\partial x} + kg_{k-1} \frac{\partial g_0}{\partial x} = 0, \quad k \geq 0.
\]

(3.5)

The second equation of system (3.2) takes the form

\[
\frac{\partial V}{\partial t} + \frac{\partial g_1}{\partial x} = 0.
\]

This equation follows from (3.3). But it should not be added to system (3.5) since it coincides with the equation of continuity \( (k = 0) \).

It is well known that a simple exchange of velocities occurs during an elastic impact of two equal particles as if the particles do not “notice” each other. From this standpoint the medium will be collisionless, and instead of the nonlinear system (3.5) we obtain a linear system of equations

\[
\frac{\partial g_k}{\partial t} + \frac{\partial g_{k+1}}{\partial x} = 0, \quad k \geq 0.
\]

However, if we continue to distinguish between equal particles when considering the elastic collision, their statistics will be described by the nonlinear system (3.5).

The potential in the form of the Dirac \( \delta \)-function determines the law of elastic collision of two particles. If this generalized function is replaced with the more generalized function \( \mu \delta(x - y) \) having an arbitrary positive coefficient, this potential will again define the elastic collision. However, in the continuous case the situation is different: the last term in the equations of motion (3.5) will contain the multiplier \( \mu \), which will lead to a change in the dynamics of the medium consisting of colliding particles. Still, for \( \mu \neq 0 \) the substitution \( g_k \mapsto g_k/\mu \), \( k \geq 0 \) leads again to equation (3.5).

Equations (3.5) were derived by Benney as far back as 1973 from a system of equations of the two-dimensional flows of a non-viscous incompressible fluid with a free surface in the gravitational field to a long wave approximation [11]. Afterwards, system (3.5) was obtained from the Vlasov kinetic equation [12, 13]. Moreover, its full integrability as an infinite-dimensional Hamiltonian system [12, 14, 15] was established.

In non-trivial cases the conditions for solvability of the infinite chain (3.2) reduce to the solvability of the initial Vlasov equation. Let

\[
g_0^0(x), g_1^0(x), g_2^0(x), \ldots
\]

(3.6)
be a sequence of Cauchy data (e.g., at $t = 0$) for unknown functions $\{g_k\}_0^\infty$. Assume that for all $x$ this sequence will be positive. Then (by Hamburger’s theorem) for all $x$ a non-decreasing function $u \mapsto \sigma(u, x)$ will be found such that

$$
\int_{-\infty}^{\infty} u^k d\sigma(u, x) = g^0_k(x)
$$

(see, e.g., [11]). The function $\sigma$ is almost everywhere differentiable:

$$
\frac{\partial \sigma}{\partial u} = \rho^*_0(x, u) \geq 0,
$$

where $\rho^*_0(x, u) = \rho^*(x, u, 0)$.

Next we shall consider $\rho^*_0$ as the Cauchy condition for the initial Vlasov equation. If $\rho^*_0 \in L^p$, then we should seek a generalized solution to this equation, and if $\rho^*_0$ is a smooth function, we need a classical solution. The conditions for existence and uniqueness of such solutions to equation (1.2) can be found in [8, 10]. If there exists a corresponding solution, there is also a solution of the infinite chain of equations (3.2), which is given by formulae (3.1). The potential $V$ is calculated using formula (3.4).

We end this section with a discussion of stationary solutions of system (3.2) for periodic boundary conditions on the assumption that the potential $K$ depends on the distance $|x - y|$ and has no singularity. We show that then this system admits a simple particular solution

$$
g_k = c_k = \text{const.} \tag{3.7}
$$

Of course, the set of numbers $\{c_k\}_0^\infty$ must be a positive sequence.

As a matter of fact, we only need to show that $V = \text{const}$ if $g_0 = \text{const}$. Indeed, according to (3.4) and (2.13),

$$
\frac{\partial V}{\partial x} = g_0 \int \frac{\partial K}{\partial x} dy = -g_0 \int \frac{\partial K}{\partial y} dy = 0.
$$

However, not all equilibrium solutions (3.7) are stable. This depends on the kind of potential $K$. We shall return to this issue in Section 8.

4. THE ENERGY INTEGRAL

By virtue of the conservative property, the infinite chain of equations of motion admits conservation laws. A direct consequence of the equation of continuity is the law of conservation of mass

$$
\int \rho(x, t) \, dx = m = \text{const.}
$$

In addition, the law of conservation of momentum is deduced from the Euler equation (2.8)

$$
\int \rho v_i \, dx = \text{const}, \quad 1 \leq i \leq n.
$$

To deduce it, one needs to prove an equality that is not quite obvious

$$
\int \rho \frac{\partial V}{\partial x_i} \, dx = 0.
$$

Its proof uses identity (2.13).

The full energy of the system is also conserved. But here one needs to precisely determine kinetic and potential energy and try to express these quantities in terms of Eulerian variables. Assume that

$$
T(t) = \frac{1}{2} \int \int u^2 \rho^*(x, u, t) \, dx \, du \tag{4.1}
$$
is a kinetic energy of the continuum and

\[ W(t) = \frac{1}{2} \iiint \rho^*(x, u, t) \rho^*(y, w, t) K(x, y) \, dx \, du \, dw \]  

(4.2)
is the total potential energy of interaction. Of course, the integrals (4.1) and (4.2) are assumed to converge and to be smooth time functions. If we omit the multiplier \( \frac{1}{2} \) in formula (4.2), the potential energy of interaction of each pair of particles will be taken into account twice.

If \( K(x, y) = K(y, x) \), then the full energy of the system is conserved:

\[ T(t) + W(t) = \text{const.} \]  

(4.3)

This assertion is proved using the Vlasov equation and the Gauss–Ostrogradsky formula (see [10]).

In view of notation (2.1),

\[ W = \frac{1}{2} \iint \rho(x, t) \rho(y, t) K(x, y) \, dx \, dy. \]  

(4.4)

It should not be supposed that

\[ T = \frac{1}{2} \int v^2 \rho \, dx, \]  

(4.5)

where \( v \) is determined from formula (2.2). This formula does not hold even for \( n = 1 \). In fact, the integral (4.1) is always larger than integral (4.5). Indeed,

\[
\begin{align*}
\iint u^2 \rho^* \, dx \, du & - \int \left[ \frac{u \rho^*}{\rho} \right]^2 \rho \, dx \\
& = \int \frac{1}{\rho} \left\{ \iint u^2 \rho^* \, du \, \rho^* \, du - \left[ \iint u \rho^* \, du \right]^2 \right\} \, dx > 0
\end{align*}
\]  

(4.6)

according to the Cauchy–Bunyakovsky inequality. In (4.6) the sign of strict inequality is used because the relation of functions \( u \sqrt{\rho^*} \) and \( \sqrt{\rho^*} \) is not constant.

We emphasize that the integral of energy (4.3) is not a consequence of only the Euler equation (2.8) and the equation of continuity. The equations of motion for a phenomenological ideal continuous medium admit an energy integral in which the kinetic energy is given by formula (4.5) rather than (4.1). In our case this conservation law is a consequence of the entire infinite chain of equations of motion. For \( n = 1 \) the integral (4.1) is obviously equal to

\[
\frac{1}{2} \iint g_2(x, t) \, dx.
\]

Since the kinetic energy is non-negative, the energy integral (4.3) can be used to investigate the stability of equilibrium states. The Lagrange–Dirichlet principle states that if the potential energy assumes a strictly minimal value, the equilibrium is stable.

**Example.** We demonstrate this principle using a simple problem of stability of the equilibrium of a one-dimensional medium with a periodic boundary condition, which was discussed at the end of Section 3. Set again \( K = \delta(x - y) \). This potential corresponds to the law of elastic impact.

First we make a general remark. Set \( u = v_0 + w, \ v_0 = \text{const} \). It is clear that \( w \) is the relative velocity of particles in a reference frame moving with velocity \( v_0 \). We confine ourselves to considering the motions of a continuous medium where the momentum of relative motion has a zero value:

\[
\iint \rho^* w \, dx \, du = 0.
\]  

(4.7)

This assumption rules out the possibility of motion of a medium with constant velocity \( v \neq v_0 \). Otherwise the equilibrium state would be obviously unstable. In view of (4.7),

\[
\frac{1}{2} \iint u^2 \rho^* \, dx \, du = \frac{mv_0^2}{2} + \frac{1}{2} \iint w^2 \rho^* \, dx \, du.
\]
In the investigation of the stability of relative equilibrium the first constant term plays no role. For elastic collisions the total potential energy (4.4) is equal to

$$W = \frac{1}{2} \int \rho^2(x, t) \, dx. \quad \text{(4.8)}$$

Let $\rho = \rho_0 > 0$ be an equilibrium value of density. Set

$$\rho(x, t) = \rho_0 + \xi(x, t),$$

where $\xi$ is density perturbation. Since the total mass of matter does not change during perturbations,

$$\int \xi(x, t) \, dx = 0. \quad \text{(4.9)}$$

The functional (4.8) assumes a stationary value at point $\rho = \rho_0$ under the restriction (4.9):

$$W = \frac{1}{2} \int \rho_0^2 \, dx + \frac{1}{2} \int \xi^2 \, dx.$$

The second term is the second variation of the functional (4.8) at point $\rho = \rho_0$. This quadratic form is obviously positive definite. This proves the stability of both the equilibrium state and stationary motions. In particular, no appreciable clusters of particles arise in a continuous medium with elastic impacts.

5. LAGRANGE’S IDENTITY

If $K(\cdot)$ is a homogeneous function of degree $s$, then

$$\ddot{I} = 4T - 2sW, \quad \text{(5.1)}$$

where

$$I = \iint x^2 \rho^*(x, u, t) \, dx \, du = \int x^2 \rho(x, t) \, dx$$

is the moment of inertia of a continuum of particles relative to the origin of the reference frame ($x^2 = \Sigma x_i^2$) and $T$ and $W$ are the kinetic energy and potential energy, respectively, which are determined by formulae (4.1) and (4.4). In view of the energy integral

$$T + W = h$$

the equality (5.1) is represented as

$$\ddot{I} = 4h - 2(s + 2)W. \quad \text{(5.2)}$$

Formula (5.1)(as well as (5.2)), an immediate consequence of the Vlasov kinetic equation, was obtained in [10]. The new observation is that the variables appearing in (5.2)(moment of inertia and total potential energy) admit a simple and natural representation in terms of Eulerian variables.

A number of interesting consequences can be derived from formula (5.2). For instance, let $s = -2$: the force of interaction decreases as the cube of the distance between particles. Then $\ddot{I} = 4h = \text{const}$. Hence, if $h > 0$, the cloud of particles flies apart, and if $h < 0$, a collapse occurs: at some time this cloud collapses in its center of mass.

We give two more examples.

Example 1. Consider the case where the interaction is absent: $K = 0$. This function will be homogeneous with any degree of homogeneity $s$. Since $W = 0$, the Lagrange identity (5.1) will have the form $\ddot{I} = 4T$ for all $s$. By the energy conservation law $T = h = \text{const}$, with $h > 0$, if $\rho^* > 0$ on a set of positive measure. Thus, $I(t) \to \infty$ as $t \to \pm \infty$, and the particles of collisionless gas fly apart to infinity.
Example 2. Let now $K = \delta(x - y)$. This is a homogeneous generalized function with degree of homogeneity $s = 0$. The Lagrange formula (5.1) has the same form as in example 1. However, it has a different content as in this case the kinetic energy is no longer conserved. By (5.2),

$$\ddot{I} = 4(h - \int \rho^2(x, t) \, dx).$$

For gravitational attraction, when $s = -1$, formula (5.1) takes the form

$$\ddot{I} = 4T + 2W.$$  \hspace{1cm} (5.3)

Consider a particular case of motion in three-dimensional Euclidean space, where the continuous medium with gravitational attraction of particles rotates as a rigid body with constant angular velocity $\omega$ relative to a space-fixed axis. Such motions will be relative equilibria. Evidently, for such motions $I(t) = \text{const}$. Therefore, (5.3) yields the condition of relative equilibrium

$$2T + W = 0.$$  \hspace{1cm} (5.4)

Now recall that the kinetic energy $T$ exceeds the integral (4.5) whose value at relative equilibrium is equal to $J\omega^2/2$, where $J$ is the body’s moment of inertia relative to the axis of revolution. Therefore, (5.4) leads to the inequality

$$J\omega^2 + W < 0.$$  \hspace{1cm} (5.5)

Recall that for the gravitational attraction $W < 0$.

It is useful to compare condition (5.5) with the condition for the equilibrium of a homogeneous rotating fluid whose particles are attracted by the law of gravity. It was obtained by Poincaré [12] using the equations of motion for an ideal fluid and has a different form

$$W + \frac{J\omega^2}{2} = \frac{3}{5} U_0 \Lambda,$$

where $U_0$ is the value of the function

$$V + \frac{\omega^2}{2} (x_1^2 + x_2^2)$$  \hspace{1cm} (5.6)

at the boundary of the body (rotation occurs about the axis $x_3$; at all points of the boundary of the rotating fluid the function (5.6) is constant) and $\Lambda$ is the volume of the rotating fluid. We emphasize that the Lagrange identity was not taken into account in deducing the Poincaré condition. For relative equilibria of a discrete set of gravitating masses we have an exact equality instead of inequality (5.5).

(5.5) gives the limit value of angular velocity:

$$\omega^2 < -\frac{W}{J}.$$  \hspace{1cm} (5.7)

This formula can be compared with the Poincaré limit for a homogeneous rotating fluid:

$$\omega^2 < 2\pi \gamma \rho,$$  \hspace{1cm} (5.8)

where $\gamma$ is a gravitational constant and $\rho$ is a constant density. The equilibrium configuration of the rotating fluid is flattened at the poles. Therefore, if in (5.7) $W$ is replaced with the gravitational energy of a homogeneous ball and $J$ with its moment of inertia, then the relation $-W/J$ will only increase. Then we obtain from (5.7) the following rough estimate

$$\omega^2 < \frac{5}{2} \pi \gamma \rho,$$  \hspace{1cm} (5.9)

This estimate does not differ greatly from (5.8).

Of course, one should bear in mind that inequality (5.7) determines the condition for the equilibrium of another, more complicated system, which is governed by an infinite chain of equations of motion. Therefore, comparison between formulae (5.8) and (5.9) reveals only a formal resemblance between them.
6. DISSIPATIVE PROPERTIES OF THE EQUATIONS OF MOTION

Can the solutions to the equations of motion for a continuum of interacting particles exhibit irreversible behavior? Seemingly, there is an evident negative answer to this question as conservative interaction underlies the model considered here, and the infinite chain of equations of motion admits the law of conservation of energy. However, contrary to the concepts based on the well-known properties of classical equations for ideal media, the model in question should actually be regarded as irreversible.

To make this clear, we first consider the simplest case where the interaction is absent ($K = 0$). Then the Vlasov kinetic equation is replaced with the more simple Liouville equation. For simplicity’s sake we take the one-dimensional case where the particles of a collisionless gas are located within the section

$$I = \{0 \leq x \leq l\}$$

and are elastically reflected from its ends. The more general problem of evolution of a collisionless gas in a rectangular parallelepiped with mirror walls is technically no more complicated.

Let $\rho^*_0(x, u)$ be the density of distribution at the initial time. It is convenient to pass on to the covering

$$\mathbb{T}^1\{\varphi \mod 2\pi\} \to I, \quad (6.1)$$

by specularly reflecting the section relative to one of the ends and by identifying the ends of the doubled section obtained. As a result, instead of the oscillatory motion of a point along the section we will obtain rotational motion along the circumference in one direction.

The covering (6.1) is represented by the following explicit formulae:

$$\varphi = \frac{\pi x}{l}, \quad \text{when } x \text{ increases from } 0 \text{ to } l$$

(the point moves to the right),

$$\varphi = 2\pi - \frac{\pi x}{l}, \quad \text{when } x \text{ decreases from } l \text{ to } 0$$

(the point moves to the left).

The angular velocity of motion of a particle along the circumference is constant:

$$\dot{\varphi} = \frac{\pi u}{l}.$$

The regularization (6.1) was considered already by Poincaré [13]. The multidimensional case is discussed in [14].

Now we lift the initial density $\rho^*_0 : I \times \mathbb{R} \to \mathbb{R}$ to the function $\tilde{\rho}^*_0 : \mathbb{T}^1 \times \mathbb{R} \to \mathbb{R}$ (under reflections from the end of the section $I$ and the involution $u \mapsto -u$ the function $\tilde{\rho}^*_0$ does not change). The periodic function $\tilde{\rho}^*_0$ of the coordinate can be expanded in its Fourier series:

$$\sum_{-\infty}^{\infty} \rho_m(u)e^{im\varphi}. \quad (6.2)$$

Its coefficients are simply calculated from the initial density:

$$\rho_m = \frac{1}{2l} \left[ \int_0^l \rho^*_0(x, u)e^{i\pi mx/l} \, dx + \int_0^l \rho^*_0(x, -u)e^{-i\pi mx/l} \, dx \right].$$

**Theorem 1.** Let $g(u)$ be a measurable function and all functions $g\rho_m$, $m \in \mathbb{Z}$ be summable. If

$$\sum_{-\infty}^{\infty} \int |g\rho_m| \, du < \infty,$$

REGULAR AND CHAOTIC DYNAMICS Vol. 16 No. 6 2011
then
\[ \int_{-\infty}^{\infty} g(u) \rho^*(x, u, t) \, du \to \int_{-\infty}^{\infty} g(u) \rho_0(u) \, du \]
as \( t \to \pm \infty \) uniformly in \( x \in I \).

Recall that
\[ \rho_0(u) = \frac{1}{2l} \left[ \int_0^l \rho_0^*(x, u) \, dx + \int_0^l \rho_0^*(x, -u) \, dx \right] \]
is a free coefficient in the Fourier series (6.2). This function is obviously even. As shown in [14], \( \rho_0 \) is the weak limit of the density \( \rho^* \), a solution of the kinetic Liouville equation as time grows infinitely.

**Corollary 1.** If
\[ \sum \int |\rho_m(u)| \, du < \infty, \quad (6.3) \]
then \( \rho(x, t) \to m/l \) as \( t \to \pm \infty \) uniformly in \( x \).

This assertion was proved earlier in [15]. Adding an arbitrary function \( g \) does not create any new difficulties.

Consider now the behavior of velocity moments (3.1) as time grows infinitely.

**Corollary 2.** If
\[ \sum \int |u^k \rho_m(u)| \, du < \infty, \quad (6.4) \]
then
\[ f_k(x, t) \to \frac{l^k}{m} \int_{-\infty}^{\infty} u^k \rho_0(u) \, du \]
as \( t \to \pm \infty \) uniformly in \( x \).

Since \( \rho_0 \) is an even function, \( \lim f_k = 0 \) for odd \( k \); while if \( k \) is even, \( \lim f_k > 0 \).

**Corollary 3.** If (6.3) is satisfied and
\[ \sum \int |u \rho_m(u)| \, du < \infty, \]
then
\[ v(x, t) \to 0, \quad (6.5) \]
as \( t \to \pm \infty \) uniformly in \( x \in I \).

The relation (6.5) looks surprising since no friction is introduced in the problem statement. This circumstance radically distinguishes the collisionless gas from the classical ideal gas, which will execute sustained oscillations.

The condition of the form (6.3) is known to be equivalent to the absolute convergence of the Fourier series (6.2). Probably it can be weakened to ensure stabilization of the density of the collisionless gas as time grows infinitely. However, weakening condition (6.3), we can lose the property of uniform convergence.

In the presence of interaction the general picture remains the same. But its analysis becomes technically much more complicated. We point out two essential things. First, the definition of weak
convergence of solutions to the Vlasov kinetic equation needs to be strengthened by replacing the
ordinary time limit with the limit of arithmetic averages (in the Cesàro sense). Second, the condition
of weak convergence in the Cesàro sense is closely related to the existence of invariant countably-
additive measures for dynamical systems in infinite-dimensional spaces, which is generated by the
Vlasov kinetic equation. For details, see [10].

Example. Consider the case where
\[ K(z) = \frac{kz^2}{2}, \quad k = \text{const} > 0. \]

We have a continuum of particles connected to each other by means of elastic springs with elasticity
coefficient \( k \). The center of mass of this system moves uniformly and in a straight line. Let us
introduce an inertial reference frame whose origin coincides with the center of mass of a continuum
of oscillators. It is straightforward to verify that in this system the Vlasov equation becomes the
Liouville kinetic equation for a usual harmonic oscillator:
\[
\frac{\partial \rho^*}{\partial t} + \sum \frac{\partial \rho^*}{\partial x_i} u_i - \frac{k}{\partial u_i} x_i = 0
\]
(see [10, §2]). It is easily solved. We write in explicit form its solution for \( n = 1 \). Let \( f(x, u) \) be the
initial density (the Cauchy datum for \( t = 0 \)). Then
\[
\rho^*(x, u, t) = f(x \cos kt - \frac{u}{k} \sin kt, kx \sin kt + u \cos kt).
\]

(6.6)

It is clear that this function is periodic in \( t \) and therefore has no ordinary limit as \( t \to \infty \)
(of course only if \( f \neq \text{const} \)). The density of the continuous medium of the oscillators \( \rho(x, t) \) is
also \( 2\pi/\sqrt{k} \)-periodic in \( t \) and is therefore not stabilized either as \( t \to \infty \).

By the way, the elastic interaction is actually the only exception when the Liouville equations
have no weak limit as time grows infinitely. The only way out is to introduce additional averaging:
we replace the ordinary convergence in time in the Cesàro sense. The introduction of additional
averaging is a standard technique in statistical mechanics.

In our case the density (6.6) weakly converges in the Cesàro sense to the measurable function
\[
g(\xi), \quad \xi^2 = \frac{u^2}{2} + \frac{kx^2}{2}.
\]

In its turn the density \( \rho \) in the configuration space weakly converges to
\[
\bar{\rho}(x) = \int_{-\infty}^{\infty} g(\xi) d\xi.
\]

Under some additional conditions for the function \( g \) the density \( \bar{\rho}(x) \) will decrease infinitely
as \( |x| \to \infty \). For example, if \( g(\xi) = c \exp(-\beta \xi^2), \beta > 0 \), then \( \bar{\rho} \) will be proportional to the normal
distribution density.

In the most general case the weak Cesàro limit of solutions to the Vlasov kinetic equation will be
a stationary (possibly generalized) solution to the same equation. The problem of their description
is quite non-trivial. The first simple observations along these lines are contained in [5].

For \( n = 1 \) the stationary solutions of the Vlasov equation satisfy the non-linear functional integral
equation
\[
\rho^*(x, u) = f\left(\frac{u^2}{2} + \iint K(x, y)\rho^*(y, w) dy dw\right).
\]

(6.7)

Here \( f \) is an a priori unknown function of one variable. From (6.7) we obtain the iteration formula
\[
\rho^* = f\left(\frac{u^2}{2} + \iint Kf\left(\frac{w^2}{2} + \iint \ldots \right) dy dw\right).
\]
The problem rests on the proof of convergence of these iterations and the determination of the degree of smoothness of the sought-for function.

For example, let $K$ be the potential of the gravitational interaction and $f(z) = \alpha \exp(-\beta z)$, $\alpha, \beta = \text{const} > 0$ (Gibbs distribution). As is shown in [16], such a stationary solution of the Vlasov kinetic equation does not exist in the entire phase space. On the other hand, for a specially chosen function $f$ equation (6.7) admits globally defined solutions (see [17–19]).

7. STATIONARY SOLUTIONS OF THE VLASOV EQUATION

The question of the existence and smoothness of solutions to equation (6.7) is quite complicated. We shall discuss it for the case with periodic boundary conditions where the configuration space is an $n$-dimensional torus

$$T^n = \{x_1, \ldots, x_n \mod 2\pi\}$$

and the kernel $K$ in the Vlasov equation contains a small parameter $\varepsilon$ as a multiplier. We shall seek stationary periodic solutions to this equation, which can be represented as a power series in $\varepsilon$ (even if it is formal):

$$\rho^* = f_0(x, u) + \varepsilon f_1(x, u) + \ldots$$

(7.1)

All functions $f_k: T^n \times T^n \to \mathbb{R}$ ($k \geq 0$) are assumed to be smooth (infinitely differentiable). Substituting (7.1) into the Vlasov equation gives an infinite chain of equations for a sequential calculation of the coefficients $f_0, f_1, \ldots$.

By the way, instead of introducing the small parameter $\varepsilon$, we can set $\varepsilon = 1$ in (7.1). Then the series (7.1) will correspond to one of possible methods of sequential approximations for finding an exact solution of the stationary Vlasov equation. In general it is equivalent to the well-known Hilbert method of solving the Boltzmann kinetic equation (see [20]).

Recall our assumption that the kernel $K: T^n \times T^n \to \mathbb{R}$ does not change when the variables are rearranged. Set

$$W(x) = \int_{T^n} K(x, y) dy.$$  

We shall regard this potential as a smooth function. At any rate, we shall assume its Fourier coefficients to be well defined

$$w_k = \frac{1}{(2\pi)^n} \int_{T^n} W(x) e^{ik(x)} dx, \quad k \in \mathbb{Z}^n.$$  

If the kernel is periodic and depends only on the difference $|x - y|$, then $w_k = 0$ for all $k \neq 0$. This case is of no interest to us.

We introduce the resonance set $P \subset \mathbb{R}^n = \{u\}$ consisting of hyperplanes

$$(k, u) = 0,$$

(7.2)

with $w_k \neq 0$. In a typical case, of course, it fills the velocity space everywhere densely. Thus, we shall assume that the closure $P$ coincides with the entire space $\mathbb{R}^n = \{u\}$.

We shall sequentially seek coefficients of series (7.1). The functions $f_0$ and $f_1$ satisfy the following equations:

$$\frac{\partial f_0}{\partial x_i} u_i = 0,$$

(7.3)

$$\frac{\partial f_1}{\partial x_i} u_i - \frac{\partial f_0}{\partial u_j} \frac{\partial}{\partial x_j} \int K(x, y) f_0(y, w) dy dw = 0.$$

(7.4)

We start by solving the first equation using the Fourier method. Let

$$f_0 = \sum f^{(k)}(u)e^{ik(x)}, \quad k \in \mathbb{Z}^n.$$
Then (7.3) gives the following relations

\[(k, u) f^{(k)}(u) = 0, \quad k \in \mathbb{Z}^n.\]

Since the functions \(f^{(k)} : \mathbb{R}^n \to \mathbb{R}\) are continuous, they are identically equal to zero for \(k \neq 0\). Hence, \(f_0 = f^{(0)}\), therefore \(f_0\) does not depend on the angular variables \(x_1, \ldots, x_n\).

But then

\[
\iint K(x, y) f_0(w) \, dy \, dw = m \frac{m}{(2\pi)^n} W(x),
\]

where

\[
m = \iint f_0(u) \, dx \, du \tag{7.5}
\]

is the mass of a gas for \(\varepsilon = 0\). Of course, we assume that \(0 < m < \infty\). Hence, equation (7.4) takes the form

\[
\frac{\partial f_1}{\partial x_i} u_i = m \frac{\partial f_0}{\partial u_j} \frac{\partial W}{\partial x_j}. \tag{7.6}
\]

It can also be solved by the Fourier method. It is equivalent to an infinite chain of relations

\[
(k, u) F_k(u) = m \frac{(2\pi)^n}{\partial x} \left( k, \frac{\partial f_0}{\partial u} \right) w_k,
\]

where \(F_k\) are the Fourier coefficients of \(f_1\) as a periodic function of \(x\).

Let \(u \in P\). Then \((k, u) = 0, \) while \(w_k \neq 0\). Hence,

\[
\left( k, \frac{\partial f_0}{\partial u} \right) = 0.
\]

Since \(k \neq 0,\)

\[
\frac{\partial f_0}{\partial u} = \lambda u \tag{7.6}
\]

on the hyperplane \((k, u) = 0\). Since \(P\) is everywhere dense in \(\mathbb{R}^n = \{u\}\), the relation (7.6) is satisfied identically. But then \(f_0\) is an energy function:

\[
f_0 = \Phi \left( \frac{u_1^2 + \ldots + u_n^2}{2} \right) \tag{7.7}
\]

The function \(s \mapsto \Phi(s)\) is a smooth function of one variable for \(s > 0\). The only condition is the convergence of the integral (7.5).

Further, the equation in \(f_1\) becomes:

\[
\frac{\partial f_1}{\partial x_i} u_i = m \frac{(2\pi)^n}{\Phi} \frac{\partial W}{\partial x_i} u_i.
\]

Hence

\[
f_1 = m \frac{(2\pi)^n}{\Phi} \Phi' W + \alpha(u) \tag{7.8}
\]

The function \(\alpha\) should be chosen from the condition

\[
\iint f_1 \, dx \, du = 0. \tag{7.9}
\]

This means that for all \(\varepsilon\) the mass of a gas is the same and equals the integral (7.5). In particular, the function \(\alpha\) actually depends not on all components of velocity \(u_i\) but only on the sum of their squares.

Let the solution of the Vlasov equation weakly converge (in the Cesàro sense) to density (7.1) and the coefficients \(f_0\) and \(f_1\) be determined by the relations (7.7) and (7.8). We shall consider
this density up to terms of order $\varepsilon^2$. It is the density of a stationary (equilibrium) state. Since this function contains only the squares of velocities $u_1^2, \ldots, u_n^2$, the average velocity $v$ vanishes. Therefore the continuous medium in an Eulerian description is at rest. Its density in configuration space reads

$$\rho(x) = c_0 + c_1 W(x) + o(\varepsilon),$$

where $c_0$ and $c_1$ are some constants with $c_0 > 0$. Therefore, for small $\varepsilon$ the density is almost constant.

By virtue of formulae (7.7), (7.8) and condition (7.9) the stress tensor

$$P_{ij} = \int u_i u_j \rho^* \, du$$

is spherical: $P_{ij} = p \delta_{ij}$, where

$$p = \int u_1^2 \rho^* \, du = \ldots = \int u_n^2 \rho^* \, du. \quad (7.10)$$

It is well known that the pressure exerted on an area with unit normal $w$ is given by the formula

$$\int (u, w)^2 \rho^* \, du$$

(see [6, 14]). In view of formulae (7.10) this integral is equal to $p$. Thus, in the state of statistical equilibrium we have the relations of an ideal gas. In particular, the Pascal law holds. For $\varepsilon = 0$ the pressure is constant, and for small $\varepsilon$ it weakly changes from point to point.

We conclude this section with some remarks.

1°. If in formula (6.7) $K$ is replaced with $\varepsilon K$ and the right-hand side is expanded in terms of $\varepsilon$, then two terms of this expansion will be determined exactly by formulae (7.7) and (7.8) (with the obvious replacement of $\Phi$ with $f$). Thus, our reasoning gives a partial justification for formula (6.7) in the multidimensional case.

2°. If in the periodical case the kernel $K$ depends only on the distance, the function $W$ will be a constant. Adding a suitable constant to the function $K$, we can always regard this constant as equal to zero. Indeed, the replacement of $K$ with $K + c$, $c = \text{const}$ does not change the original Vlasov equation. But then, by (6.7), the stationary solutions of the Vlasov equation contain the function

$$f\left(\frac{u_1^2 + \ldots + u_n^2}{2}\right).$$

The function of one variable $f$ can be an arbitrary measurable function; the only condition is the convergence of the integral (7.5). This remark is in contrast with the exaggerated attention to the Gibbs canonical distribution.

3°. The method of seeking stationary solutions to the Vlasov kinetic equation in the form of a power series in a small parameter is a generalization of the well-known Poincaré method that involves finding the first integrals of the Hamilton equations, which differ little from completely integrable equations (see [21, 22]).

8. TURBULENCE MODELING

The onset of turbulence is usually associated with the loss of stability of a laminar flow at high Reynolds numbers. On the other hand, turbulent flows are described using the Navier–Stokes equations with additional averaging and closure operations. However, these new assumptions are so essential that one can safely forget about the classical hydrodynamics of laminar flows.

Physically speaking, the approach based on the use of kinetic equations is more preferable and fundamental to the study of turbulence. It is considered that the hydrodynamic equations themselves (including viscous equations) can be obtained by methods of statistical mechanics. The direct use of Boltzmann-type kinetic equations for modeling of turbulent flows is no longer something unusual. For these purposes, the Kac kinetic equation is used in [23]. Uhlenbeck called the Kac model a caricature of a real gas, though. We shall use the Vlasov equation as the initial point.
The central feature of turbulence theory is the loss of stability of a homogeneous flow when all particles move in straight lines with constant velocity. From the molecular-kinetic point of view, the instability of such flows means the formation of individual molecular clusters. We investigate the problem of stability of one-dimensional homogeneous motion. In doing so, we are, of course, aware that this is a model situation.

Can we speak of one-dimensional turbulence at all? If we consider a non-compressible fluid, the answer is clearly negative. Conversely, the one-dimensional flows of a compressible fluid or gas can lose stability: there are appreciable velocity and density fluctuations possible.

In the theory which we are developing based on the Vlasov kinetic equation, the stability condition depends on the property of the potential for interaction between particles. First, we consider the simpler case where the continuum of interacting particles uniformly rotates around the circumference.

Thus, let the circumference \( \{x \mod 2\pi\} \) be a configuration space and the potential \( K(x, y) \) be an even periodic function of the difference \( x - y \). Passing to the moving reference frame that rotates uniformly with an angular velocity equal to that of the uniform rotation, we reduce the problem at hand to the problem of stability of a homogeneous equilibrium state where the density of the medium is constant.

In order to solve it, we shall make use of the Lagrange–Dirichlet energy criterion: an equilibrium is stable if and only if the potential energy (4.4) in this equilibrium has a strict minimum. We have already used this principle in Section 4 for a particular problem where \( K(z) = \delta(z) \). In the exact sense, there can be no strict minimum here as the equilibrium positions arising from each other by turns of the circumference form a whole one-parameter family. However, this difficulty is easily surmounted by choosing perturbations.

Let
\[
\rho(x, t) = \rho_0 + \xi(x, t), \quad \rho_0 = \text{const} > 0,
\]
and \( \xi \) be the density perturbation. We shall proceed from the assumption that the total mass does not vary:
\[
\int_0^{2\pi} \xi(x, t) \, dx = 0. \tag{8.1}
\]

Further,
\[
\frac{1}{2} \iint (\rho_0 + \xi(x, t))(\rho_0 + \xi(y, t)) K(x - y) \, dx \, dy \\
= \frac{\rho_0^2}{2} \iint K \, dx \, dy + \frac{\rho_0}{2} \iint (\xi(x, t) + \xi(y, t)) K(x - y) \, dx \, dy \\
+ \frac{1}{2} \iint \xi(x, t)\xi(y, t) K(x - y) \, dx \, dy. \tag{8.2}
\]
The first term is equal to zero in view of identity (2.13), and the second term also goes to zero in view of (2.13) and assumption (8.1).

We expand the potential \( K \) into a Fourier series:
\[
K(x - y) = \sum k_n e^{in(x - y)}.
\]
Since \( K(\cdot) \) is an even function, all \( k_n \) are real numbers. Further, let
\[
\xi(x, t) = \sum' \xi_p(t) e^{ipx}, \quad \xi(y, t) = \sum' \xi_p(t) e^{ipy}.
\]
The prime denotes omission of the term with \( p = 0 \) (in view of assumption (8.1)). Then the second variation of the potential energy (the third term in (8.2)) is equal to
\[
\sum' k_m \xi_m \xi_m'.
\]
Since $\xi_m \overline{\xi}_m \geq 0$, the criterion of positive definiteness of the second variation is reduced to an infinite series of inequalities

$$k_m > 0, \quad m \neq 0. \quad (8.3)$$

We emphasize that the sign of the “free” constant $k_0$ has no meaning. In fact, this has been obvious from the outset: the potential is defined up to an additive multiplier.

If $k_p < 0$, the perturbation of the p-mode will grow exponentially fast in time. If $k_p = 0$, condition (8.3) is also violated. However, the p-mode can only grow linearly in time. But this conclusion only holds for linearized equations of motion.

**Example 3.** Set

$$K = \sin^2(x - y). \quad (8.4)$$

Then

$$K(z) = \frac{1}{2} - \frac{\cos 2z}{2},$$

therefore, condition (8.3) is violated: the second mode grows exponentially.

If we reverse the sign of the potential (8.4), all Fourier coefficients in (8.3) will be non-negative. This does not yet mean the presence of stability (in the Lagrange–Dirichlet sense), but at any rate the instability will not exhibit “explosive behavior”.

Now consider the more “realistic” problem of stability of the one-dimensional flow of a homogeneous continuum of particles in a straight line. In the analysis of the stability conditions we should replace the Fourier series with Fourier integrals, taking extra precautions.

We set again

$$\rho(x, t) = \rho_0 + \xi(x, t), \quad \rho_0 = \text{const} > 0,$$

$$\int_{-\infty}^{\infty} \xi(x, t) \, dx = 0. \quad (8.5)$$

The potential energy can again be represented as (8.2). As a rule, the first term in (8.2) becomes infinite. But there is nothing “tragic” about it for the following reasons.

1°. By subtracting a suitable constant from the potential $K$, one can often arrive at the following equality

$$\int_{-\infty}^{\infty} K(z) \, dz = 0.$$

Then the first term in (8.2) is also equal to zero.

2°. Ultimately our interest is in the total force rather than potential. Therefore, the additive term in the expression for the potential energy is irrelevant.

We formulate conditions under which the second term in (8.2) goes to zero. Set

$$\hat{\xi}(x, t) = \int_{0}^{x} \xi(x, t) \, dx.$$

By virtue of convergence of the improper integral (8.5), $\hat{\xi}(x, t)$ tends to some constants as $x \to \pm \infty$. Further,

$$\int_{-\infty}^{\infty} \xi(x, t) K(x - y) \, dx = K(x - y) \hat{\xi}(x, t) \bigg|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} K_y \hat{\xi}(x, t) \, dx,$$

$$\int_{-\infty}^{\infty} \xi(y, t) K(x - y) \, dy = K(x - y) \hat{\xi}(y, t) \bigg|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} K'_y \hat{\xi}(y, t) \, dy. \quad (8.6)$$
We shall assume that the interaction potential $K$ tends to zero at infinity. Then the integrated terms in (8.6) are equal to zero. It remains to apply the simple identity (2.13).

Thus, the extreme properties of the total potential energy $W$ are determined by the third term in (8.2). Set

$$K(x - y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} k(\lambda)e^{i\lambda(x-y)} d\lambda. \tag{8.7}$$

Since the function $K(\cdot)$ assumes real values and is even, $k(\lambda) = k(-\lambda) = \overline{k}(\lambda)$. Hence, all quantities $k(\lambda), \lambda \in \mathbb{R}$ are also real.

Further,

$$\int \int \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} k(\lambda)e^{i\lambda(x-y)} d\lambda \right] \xi(x, t)\xi(y, t) dx \, dy = \frac{1}{2\pi} \int_{-\infty}^{\infty} k(\lambda) \left[ \int_{-\infty}^{\infty} \xi(x, t)e^{i\lambda x} dx \right] \left[ \int_{-\infty}^{\infty} \xi(y, t)e^{-i\lambda y} dy \right] d\lambda = \frac{1}{2\pi} \int_{-\infty}^{\infty} k(\lambda) \eta(-\lambda, t)\eta(\lambda, t) d\lambda,$$

where $\eta$ is the inverse Fourier transform of the function $\xi$. It is clear that $\eta(-\lambda) = \overline{\eta}(\lambda)$ and $\eta\overline{\eta} \geq 0$. According to (8.5), $\eta|_{\lambda=0} = 0$.

Thus, the second variation of the total potential energy is positive definite in the class of variations satisfying (8.5) if and only if

$$k(\lambda) > 0, \quad \lambda \neq 0. \tag{8.8}$$

**Example 4.** Set again $K(z) = \delta(z)$. Then

$$k(\lambda) = \int_{-\infty}^{\infty} K(z)e^{-i\lambda z} dz = 1.$$

Consequently, condition (8.8) is obviously satisfied, which entails stability of the flow. In fact, this result was obtained in Section 4 by direct calculations.

**Example 5.** Consider the stability of a homogeneous continuum of gravitating particles on a straight line. The potential is given by the formula

$$K(z) = -\frac{\gamma}{|z|}, \quad \gamma = \text{const} > 0. \tag{8.9}$$

Unfortunately, in this case the Fourier transform (8.7) is not well defined by virtue of divergence of the improper integral due to the singularity of the potential.

Here one can reason as follows. Since the Newton law is not valid at very small distances (actually, there are doubts about the continuity of space at such distances), we replace the values of potential (8.9) in the small interval $|z| < \varepsilon$ with the large constant $-\gamma/\varepsilon$. After that, formula (8.7) will become correct with the values of the function $k(\lambda)$ being obviously negative in the interval $|\lambda| < \varepsilon_0/\varepsilon$, where $\varepsilon_0$ is a positive constant. As $\varepsilon$ decreases the negativity interval will increase, which is indicative of a strong instability of the gravitating homogeneous continuum. Of course, this reasoning cannot be regarded as completely strict. However, one should bear in mind that the problem of solvability of the Vlasov equation with potential (8.9) in the entire space still remains open (compare with [24, 25]).

The phenomenon of instability of self-gravitating continua is well known (see, e.g., the review [26] and references contained therein). There is even a term for it: gravitational turbulence. However,
the instability is usually substantiated using phenomenological (as a rule, simplified) equations for a continuous medium with the gravitational interaction of individual parts taken into account. As pointed out in Section 2, these equations differ from the exact equations of motion for a continuous medium whose evolution is governed by the Vlasov kinetic equation.

**Example 6.** Set

\[ K(z) = \frac{c}{|z|^{\alpha}}, \]

c = \text{const} \neq 0, 0 < \alpha < 1. This potential also has a singularity at zero but as opposed to gravitation, the inverse Fourier transform (8.7) is correctly defined. We have

\[
k(\lambda) = \int_{-\infty}^{\infty} \frac{c}{|z|^{\alpha}} e^{-i\lambda z} dz = c \int_{-\infty}^{\infty} \frac{\cos \lambda z}{|z|^{\alpha}} dz = \frac{c}{|\lambda|^{\beta}} \int_{-\infty}^{\infty} \frac{\cos x}{|x|^{\alpha}} dx,
\]

where \( \beta = 1 - \alpha \). In the last equation the evenness of the function \( k(\cdot) \) was used. The integral on the right-hand side is positive. Hence, if \( c > 0 \) (repulsion) we have stability, while conversely, if \( c < 0 \) (attraction), we have instability. Since \( 0 < \beta < 1 \), the Fourier transform of the function \( k(\cdot) \) is also correctly defined.

**Example 7.** In reality, molecules strongly repel each other at small distances and attract each other at large distance between them. The Lennard-Jones potential serves as a good universally accepted model:

\[
K(z) = \frac{a}{z^{12}} - \frac{b}{z^{6}},
\]

\begin{align}
\text{Figure}
\end{align}
where $a$ and $b$ are some suitable constants. Of course, for such a potential the Fourier representation has not been defined. We shall proceed as in Example 5: in the interval $|z| < \varepsilon$ we set $K(z) = ae^{-12} - b\varepsilon^{-6}$, and then we shall let $\varepsilon$ tend to zero.

The figure shows the graph of the function $k(\cdot)$ with $\varepsilon = 0.001$ for the values $a = b = 1$. It is seen that in the large frequency interval (of order $3000$) this function is positive. It can be shown that as $\varepsilon$ decreases further the range of positivity is extended, so that as $\varepsilon \to 0$ it “covers” the entire frequency axis $\mathbb{R} = \{\lambda\}$. For small but fixed $\varepsilon$ there are always negativity domains of the function $k$. This implies the onset of instability with high-frequency perturbations. However, from the pragmatical point of view taking into account the approximate meaning of the interaction potential, the equilibrium state of a homogeneous continuum of Lennard-Jones should be regarded as stable.

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REFERENCES