OPEN PROBLEM

Several problems on dynamical systems and mechanics

V V Kozlov
Steklov Mathematical Institute of Russian Academy of Sciences, 8 Gubkina Str.,
Moscow 119991, Russia

E-mail: kozlov@pran.ru

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Abstract
We discuss some open problems in the theory of dynamical systems, classical
and quantum mechanics.

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1. Integrability of geodesic flows on algebraic surfaces

Let $E$ be the three-dimensional Euclidean space with Cartesian coordinates $x = (x_1, x_2, x_3)$
and $f$ a homogeneous non-zero polynomial in $x$ of degree $m$. The geodesic flow on the
algebraic surface
\[ f(x) = c \]  
(1)
is determined by the second-order differential equation
\[ \ddot{x} = \lambda \frac{\partial f}{\partial x}, \quad \lambda = -\frac{(A \dot{x}, \dot{x})}{(f', f')}, \] \(2\)
where $(,)$ denotes the scalar product and
\[ A = \left\| \frac{\partial^2 f}{\partial x_i \partial x_j} \right\|, \quad f' = \frac{\partial f}{\partial x}. \]

By the Euler formula, $c = 0$ is the unique critical value of $f$. Hence, for $c \neq 0$ the
denominator in (2) is different from zero on the surface (1).

Equation (2) has the energy integral
\[ H = \frac{1}{2}(\dot{x}, \dot{x}). \]

If there is an additional first integral (functionally independent of $H$) then the geodesic flow is
integrable. For higher-dimensional spaces, the problem on geodesic flows on quadrics ($m = 2$)
was solved by Jacobi and Chasles.
We consider this problem for \( m \geq 3 \). This case was examined already by Riemann in his study of motion of a homogeneous liquid ellipsoid. More exactly, Riemann examined the integrability of the geodesic flow on the third-order surface

\[ x_1 x_2 x_3 = \text{const} \neq 0. \tag{3} \]

Additionally, assume that

\[ f'(x) \neq 0 \quad \text{for any } x \neq 0, \]

and that the cone \( \{ x \in E : f(x) = 0 \} \) is not just a single point (the origin). In this case the cone intersects the unit sphere

\[ x_1^2 + x_2^2 + x_3^2 = 1 \tag{4} \]

along a regular (in the real-valued sense) algebraic curve \( F \). The curve is a union of several non-intersecting ovals. On sphere (4), consider the domain \( F_+ (F_-) \) defined by the inequality \( f(x) \geq 0 \) (\( f(x) \leq 0 \)). Topologically, this domain is a union of holed discs. Estimates and comparisons of the Euler characteristics of the domains \( F_+ \) are presented in many widely known works on algebraic geometry. First, let us note that for \( c = 0 \) the problem on geodesic flow is integrable for arbitrary values of \( m \) because there is an additional first integral quadratic in velocities

\[ \Phi_1 = \left( (x, \dot{x}), [x, \dot{x}] \right). \]

Here \([ , ]\) is the vector product. For \( c \neq 0 \), integrability seems to be a rare event.

Let \( \chi \) be the Euler characteristics. I have recently shown that if \( \chi(F_+) < 0 \) (\( \chi(F_-) < 0 \)), then for all \( c > 0 \) (\( c < 0 \)) the geodesic flow on the surface (1) does not admit a first integral analytic in coordinates and velocities and is functionally independent of \( H \). In view of the Harnack algorithm for the construction of \( M \) curves, one can conclude that for any integer \( m \geq 4 \) there exists an algebraic surface of degree \( m \) such that the geodesic flow on it is non-integrable. Moreover, this property is preserved even if the coefficients of \( f \) are slightly perturbed.

Here are some problems that are not yet solved.

(a) Prove the non-integrability of the equations for geodesics on a typical cone \( \{ f = 0 \} \) in the space of arbitrary dimension. Recall that for integrability of the geodesic flow on an \( n \)-dimensional manifold \( n \) independent involutive integrals are required.

(b) Is it true that the geodesic flow on a generic third-order algebraic surface is not integrable? In particular, do I not know a rigorous proof of non-integrability for the surface (3)?

(c) The geodesic flow on the algebraic surface (1) is well defined by the coefficients of the function \( f \) and the value of \( c \). These coefficients and \( c \) can be treated as coordinates of a point in an affine space. The question is whether the conditions for the flow to be integrable are algebraic, i.e. can they be represented in the form of a set of algebraic equations in the coefficients of \( f \)?

(d) Since the right-hand sides of the equations for geodesics are quadratic in velocities, the existence of an additional integral of motion is equivalent to the existence of an integral in the form of a homogeneous polynomial in velocities and functionally independent of the energy. The additional integral of the least possible degree will be called irreducible. It is natural to ask, are there algebraic surfaces with arbitrarily large degrees of their irreducible integrals?

(e) It would be interesting to establish restrictions on the topology of algebraic surfaces of arbitrary dimension with integrable geodesic flow.

The foundations of the theory of analytic, topological and geometrical obstructions to the integrability of Hamiltonian systems can be found in the book [1].
2. Polynomial integrals

Here we consider the property of integrability of a Hamiltonian system with the Hamiltonian

\[ H = \frac{1}{2}(ay_1^2 + 2by_1y_2 + cy_2^2) + V(x_1, x_2). \]  

(5)

Here \( x_1, x_2 \mod 2\pi \) are angular coordinates and \( y_1, y_2 \) the conjugate momenta. This is a system with two degrees of freedom and the configuration space is the two-dimensional torus \( T^2 = \{ x_1, x_2 \mod 2\pi \} \). The matrix

\[ \begin{pmatrix} a & b \\ b & c \end{pmatrix} \]  

(6)

is positive definite. The potential energy \( V \) is assumed to be an analytic function, \( 2\pi \)-periodic in the coordinates \( x_1 \) and \( x_2 \).

Of course, \( H \) is a first integral of the Hamiltonian system

\[ \dot{x}_j = \frac{\partial H}{\partial y_j}, \quad \dot{y}_j = -\frac{\partial H}{\partial x_j}; \quad j = 1, 2. \]  

(7)

All known first integrals of such a system are either polynomials in momenta (with analytic and \( 2\pi \)-periodic coefficients) or functions of such polynomials. The following problem naturally arises.

(a) Prove that if system (7) has a first integral which is analytic in the phase space \( T^2 \times \mathbb{R}^2 \) and is functionally independent of the energy, then there is an independent with the energy integral which is a polynomial in momenta.

It is easy to show that the existence of a linear integral is intimately related to the existence of a ‘hidden’ cyclic coordinate, which does not enter into the expression for the potential energy, while quadratic integrals are responsible for the possibility of separation of variables.

(b) Prove that the degree of the irreducible polynomial integral of (7) does not exceed two.

In other words, given a polynomial integral independent of the energy (5), there exists a linear canonical change in variables which splits our Hamiltonian system into the direct sum of two one-dimensional systems.

This conjecture occurred as a result of two independent explorations [2,3]. In [2] integrals of degree 3 for system (7) with unit matrix (6) are studied. It is shown that there necessarily exists a linear in momenta integral of motion. A similar statement about integrals of degree four is formulated in [2] but not proved. Moreover, the proof has never been published.

In [3] problem (b) is solved in the case of polynomial integrals of arbitrary degree when the Fourier expansion of the potential energy contains only a finite number of non-zero terms. Moreover, this result is established in [3] for systems with many degrees of freedom.

Another publication on this subject is [4]. Unfortunately, no general approaches that are able to completely resolve problem (b) are known so far.

Let us formulate one more problem concerned with polynomial integrals. Consider a reversible system with two degrees of freedom whose configuration space is the two-dimensional sphere \( S^2 \). The Hamiltonian \( H = T + V \) is an analytic function in the phase space \( T^*S^2 \). The kinetic energy is a positive-definite quadratic form in momenta and \( V \) is an analytic function on the sphere.

(c) Is it true that the degree of the irreducible integral which is a polynomial in momenta does not exceed four?

Concrete examples of dynamical systems on a sphere with irreducible integrals of degrees 3 and 4 can be obtained from the Chaplygin and Kovalevskaya integrable cases of motion of a heavy top on the zero level of the angular momentum.
Problems (b) and (c) unveil interesting links between the topology of the configuration space and degrees of polynomial integrals of motion. In addition, it should be noted that at least locally irreducible polynomial integrals of arbitrarily high degree can exist.

3. Regular and chaotic quantum systems

Traditionally, quantum chaos is related to the properties of the spectrum of quantum systems. However, to begin with, quantum systems should be properly classified into regular and chaotic systems (as this is done in classical mechanics). Regular quantum systems must have a set of differential operators that are in involution with the Hamilton operator. Recall that involutive Hermitian operators give rise to conservation laws for quantum systems.

Let $M^n$ be the $n$-dimensional configuration space of a quantum system. Suppose that $M^n$ is compact and endowed with local coordinates $x = (x_1, \ldots, x_n)$ ($y = (y_1, \ldots, y_n)$ are the conjugate momenta). We start from the classical Hamiltonian

$$ H = \frac{1}{2} \sum g^{kj}(x) y_k y_j + V(x). \tag{8} $$

For simplicity below we assume all the objects to be analytic and take Planck’s constant to be 1.

According to the standard quantization routine, instead of Hamiltonian function (8) consider the Hamilton operator

$$ \hat{H} = \frac{\Delta}{2} + \hat{V}. \tag{9} $$

Here

$$ \Delta = \frac{1}{\sqrt{g}} \sum \frac{\partial}{\partial x^j} \left( \sqrt{g} g^{kj} \frac{\partial}{\partial x^k} \right) $$

is the Laplace–Beltrami operator associated with the Riemann metric on $M$, which is determined by the kinetic energy; $\hat{V}$ is the operator of multiplication by the function $V$.

Put $\partial = (\partial_1, \ldots, \partial_n)$, where $\partial_k = \frac{\partial}{\partial x^k}$ and let $\hat{F}(x, \partial)$ be a polynomial differential operator which is in involution with the Hamilton operator and defined everywhere on $M$. Preliminaries of the theory of obstructions to regular behaviour of quantum systems can be found in the survey paper [5] (see also [6–8]).

A good example is the following result: if $\chi(M^2) < 0$, then the operator $\hat{F}$ is a polynomial in $\hat{H}$ with constant coefficients [8].

Below I present some problems concerned with polynomial conservation laws in quantum mechanics. Some of the classical counterparts of these problems are already solved.

(a) Is the following statement true? Suppose that there is an operator of the general form which is in involution with the Hamilton operator $\hat{H}$ and functionally independent of it; then there exists a non-trivial polynomial operator which is also in involution with $\hat{H}$.

(b) Is it true that at least locally there exist irreducible differential polynomial operators of arbitrarily high degree that are in involution with the Hamilton operator?

(c) Consider a classical Hamiltonian system with toroidal configuration space $T^n = \{x^1, \ldots, x^n \mod 2\pi\}$. Let

$$ H = \frac{1}{2} \sum \alpha^{ij} y_i y_j + V(x). \tag{10} $$

where $||\alpha^{ij}||$ is a positive-definite matrix with constant components and $V$ is an analytic function on $T^n$. Let $\hat{H}$ be obtained via the standard quantization procedure. Is it true that the degree of the irreducible polynomial differential operator that commutes with $\hat{H}$ is less than or equal to 2? If $V$ is a trigonometric polynomial, this statement is proved in [5].
I would like to mention another problem about the quantum integrability of generalized Toda lattices. The case in point is a quantum version of the Hamiltonian system (10). Now $\|a^j\|$ is a unit matrix, $x \in \mathbb{R}^n$, and the potential energy $V$ is a finite sum of the form

$$\sum v_j \exp(a_j, x), \quad v_j \in \mathbb{R}, \quad a_j \in \mathbb{R}^n.$$ 

The necessary conditions for the existence of $n$ independent polynomial operators in involution with the Hamiltonian are indicated in [6]. Are these conditions also sufficient?

4. Two problems on instability

Suppose that

$$\dot{x} = v(x), \quad x \in \mathbb{R}^n \tag{11}$$

is an autonomous system of differential equations that has an invariant measure with smooth density;

$$\text{div} (\rho v) = 0, \quad \rho (x) > 0.$$ 

Let $x = 0$ be an equilibrium solution: $v(0) = 0$.

My student V Ten proposed a conjecture that if $n$ is odd and the equilibrium $x = 0$ is isolated, then it is unstable in the sense of Lyapunov. The conjecture has an important consequence: all isolated equilibria of stationary flows of a liquid in the three-dimensional Euclidean space are unstable.

(a) Prove Ten’s conjecture in the analytic case.

For equation (11) with an infinitely differentiable right-hand side the conjecture fails to be valid. A counterexample is given in [9]. On the other hand, in [9] Ten’s conjecture is proved for so-called semi-quasihomogeneous systems of differential equations. However, this result alone does not solve the problem.

Analogously, an isolated periodic solution of an even-dimensional analytic dynamical system with invariant measure seems to be always unstable. This observation also remains valid for isolated reducible invariant tori of odd co-dimension filled with conditionally periodic trajectories.

Suppose that system (11) admits an integral of motion $F$ such that $F(0) = 0$ and $x = 0$ is an equilibrium solution. First, assume that $x = 0$ is a non-degenerate (of the Morse type) critical point of $F$ and its index of inertia is 1. Let

$$\dot{x} = Ax + o(x);$$

where $A$ is the operator of the linearized system.

It can be proved that if $\det A \neq 0$, then $n$ is even. Moreover, the spectrum of $A$ consists of a single pair of real numbers $\pm \lambda \neq 0$ and $(n - 2)/2$ pairs of purely imaginary eigenvalues with simple elementary divisors.

This proposition gives an insight into how an equilibrium solution loses stability as positive definiteness of the integral of motion (usually the total energy) is violated. The solution $x = 0$ is isolated due to the condition $\det A \neq 0$.

Assume that the vector field $v$ and the integral $F$ depend on a parameter $\varepsilon$ and for $\varepsilon = 0$ the equilibrium is a degenerate critical point of $F$. Such values of the parameter are called bifurcation values. In the typical case, for $\varepsilon = 0$ the function $F$ can be reduced to the form (in a neighbourhood of $x = 0$)

$$\frac{1}{2} (x_1^2 + \cdots + x_{n-1}^2) + \frac{x_n^3}{3}. \tag{12}$$

(b) Is it true that an isolated equilibrium of the analytic system of differential equations (11) that admits the integral (12) is always unstable?
5. Problem of rolling motion of a disc

It is well known that a wheel has a surprising ability to roll stably when pushed and released. It seems reasonable to try to explain this phenomenon within the framework of a non-holonomic model when the velocity of the point of contact of the wheel with the plane is assumed to be zero. Let our wheel be modelled as an infinitely thin disc whose centre of mass coincides with its geometrical centre. The distribution of the mass is not assumed to possess any symmetry, i.e. the central principal moments of inertia of the disc can be different.

(a) Prove that for almost all initial conditions the disc will never fall.

So far this problem has been solved only for symmetric discs [10]. The presence of an invariant measure with smooth and everywhere positive density plays an important role. It should be noted that generally non-holonomic systems are lacking an invariant measure absolutely continuous with respect to the standard Lebesgue measure. Thus, it can happen that the equations for an asymmetric disk [10] do not have an invariant measure either.

In the case of a symmetric disc the proof is based on the regularization of the equations of motion, analysis of the phase flow on the invariant ‘falling-motion manifolds’ and application of some modifications of the Poincaré recurrence theorem.

A method similar in principle to this one has long been used in celestial mechanics for analysis of collision of gravitating particles.

In the case of a non-symmetric disc slipping on smooth ice, the equations of motion are Hamiltonian and the existence of an invariant measure is a consequence of the classical Liouville theorem. Using the method described above, it can be proved that a slipping asymmetric disc almost never falls [10].

There is also an intermediate model: the velocity of the point of contact is directed along the tangent to the disc at this point (a disc with a sharp edge on an icy plane). Such a constraint is also non-holonomic and the existence of an invariant measure has been proved only in the symmetric case.

(b) Prove that for almost all initial conditions a dynamically asymmetric disc with a sharp edge will never fall onto the ice.

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