Chaplygin’s Ball Rolling Problem
Is Hamiltonian

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EQUATIONS OF MOTION AND INTEGRALS

Consider the problem of the rolling of a dynamically balanced nonsymmetric ball across a horizontal rough plane surface (the velocity of the point of contact is zero) in a potential field of force. The projections of the motion of the ball on the principal axes associated with the ball are described by the system

\[ \begin{align*}
\dot{\mathbf{M}} &= \mathbf{M} \times \mathbf{\omega} + \mathbf{\gamma} \times \frac{\partial V}{\partial \mathbf{\gamma}}, \\
\dot{\mathbf{\gamma}} &= \mathbf{\gamma} \times \mathbf{\omega}, \\
\mathbf{M} &= \mathbf{I}_0 \mathbf{\omega} + D \mathbf{\gamma} \times (\mathbf{\omega} \times \mathbf{\gamma}), \\
D &= ma^2,
\end{align*} \]

(1)

where \( \mathbf{\omega} \) is the angular velocity vector, \( \mathbf{\gamma} \) is the vertical unit vector, \( \mathbf{I}_0 = \text{diag}(I_1^0, I_2^0, I_3^0) \) is the inertia tensor of the ball about its center, \( m \) is the mass of the ball, and \( a \) is its radius. The vector \( \mathbf{M} \) stands for the kinetic momentum of the ball about the point of contact. As was shown by S. A. Chaplygin [1], for \( V = 0 \) Eqs. (1) have the integrating factor

\[ \rho = \mu^{-1} = \frac{1}{\sqrt{1 - D(\mathbf{\gamma}, (\mathbf{I}_0 + D \mathbf{E})^{-1}\mathbf{\gamma})}} = \frac{1}{\sqrt{1 - D(\mathbf{\gamma}, \mathbf{A}\mathbf{\gamma})}}, \]

(2)

and four independent integrals

\[ \begin{align*}
F_1 &= (\mathbf{M}, \mathbf{\omega}) , \\
F_2 &= (\mathbf{M}, \mathbf{\gamma}) , \\
F_3 &= (\mathbf{\gamma}, \mathbf{\gamma}) = 1 , \\
F_4 &= (\mathbf{M}, \mathbf{M}),
\end{align*} \]

(3)

which allows us to integrate the system using the Euler–Jacobi theorem. In [1] system (1) was also integrated in terms of hyperelliptic functions.

It was shown in [2] that the problem remains integrable when the field of force \( V = (k/2)(\mathbf{I}\mathbf{\gamma}, \mathbf{\gamma}) \) considered in Brun’s problem is added, but the integrals \( F_1, F_4 \) are somewhat modified:

\[ \begin{align*}
F_1 &= (\mathbf{M}, \mathbf{\omega}) + k(\mathbf{I}\mathbf{\gamma}, \mathbf{\gamma}) , \\
F_4 &= (\mathbf{M}, \mathbf{M}) - (\mathbf{C}\mathbf{\gamma}, \mathbf{\gamma}),
\end{align*} \]

(4)

where \( \mathbf{C} = \text{diag}(c_1, c_2, c_3), c_i = k_\epsilon_{ijk}(I_j + D)(I_k + D), i, j, k = 1, 2, 3. \)
ANALOGY WITH JACOBI’S PROBLEM

For $V = 0$, Eqs. (1) can be expressed in terms of angular velocities as follows:

$$
\mathbf{I} \dot{\mathbf{\omega}} = \mathbf{I} \mathbf{\omega} \times \mathbf{\omega} + D \gamma \left( \frac{\mathbf{I} \mathbf{\omega} \times \mathbf{\omega}, \mathbf{I}^{-1} \gamma}{F} \right),
$$

$$
\dot{\gamma} = \gamma \times \mathbf{\omega}, \quad \mathbf{I} = \mathbf{I}_0 + DE.
$$

At the zero level of the area integral $(\mathbf{I}_0 \mathbf{\omega}, \gamma) = 0$, Eqs. (1) have integrals of the form

$$
(\mathbf{I} \mathbf{\omega}, \mathbf{\omega}) - D(\omega, \gamma)^2 = 2h, \quad (\mathbf{I} \mathbf{\omega}, \mathbf{I} \mathbf{\omega}) - D^2(\omega, \gamma)^2 = n, \quad (\gamma, \gamma) = 1.
$$

We can see that after the substitution

$$
\mathbf{\tilde{M}} = \mu \mathbf{I}_0 \mathbf{\omega}, \quad \mu^2 = 1 - (\mathbf{I}^{-1} \gamma, \gamma) = (\gamma, \mathbf{I}^{-1} \gamma)
$$

the integrals (4) become the integrals of Jacobi’s geodesics problem on the two-dimensional ellipsoid; the latter problem can be expressed in terms of the variables $\mathbf{\tilde{M}}, \gamma$ on the algebra $e(3)$ [3]. It is readily verified by a straightforward calculation that using the variables $\mathbf{\tilde{M}}, \gamma$ under the condition $(\mathbf{I}_0 \mathbf{\omega}, \gamma) = (\mathbf{\tilde{M}}, \gamma) = 0$, we can transform our integrals into the following ones:

$$
H = n - 2hD = \frac{(\mathbf{J} \mathbf{\tilde{M}}, \mathbf{\tilde{M}})}{(\gamma, \mathbf{J}^{-1} \gamma)}, \quad K = n = \frac{(\mathbf{J} \times \gamma)^2}{(\gamma, \mathbf{J}^{-1} \gamma)}, \quad \mathbf{J} = \mathbf{I}_0^{-1},
$$

where $H$ and $K$ are, respectively, the Hamiltonian and the Joachimsthal integral of Jacobi’s problem. Thus at the zero level of the area integral Chaplygin’s problem and Jacobi’s problem have the same invariant tori.

THE NONLINEAR BRACKET.

ISOMORPHISM WITH THE BRADEN SYSTEM

Equations (1) turn out to be Hamilton equations after the time substitution

$$
dt \rightarrow \frac{1}{\rho} dt,
$$

where $\rho$ is defined by relation (2). Indeed, by sufficiently long calculations we can verify that with respect to the new time, Eqs. (1) can be written in the Hamilton form with the nonlinear Poisson bracket

$$
\{M_i, M_j\} = \varepsilon_{ijk} \rho (M_k - g \gamma_k), \quad \{M_i, \gamma_j\} = \varepsilon_{ijk} \rho \gamma_k, \quad \{\gamma_i, \gamma_j\} = 0
$$

and the Hamiltonian

$$
H = \frac{1}{2} (\mathbf{M}, \mathbf{AM}) + \frac{1}{2} g(\mathbf{AM}, \gamma) + V(\gamma).
$$

Here, as usual, we have

$$
\omega = \frac{\partial H}{\partial \mathbf{M}}, \quad g = D(\omega, \gamma) = \frac{D(\mathbf{AM}, \gamma)}{1 - D(A \gamma, \gamma)}.
$$

Note that the Poisson structure (6) is not related to the integrability of the Chaplygin ball or of its generalizations; it is valid for an arbitrary potential $V(\gamma)$.

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The bracket (6) is degenerate and possesses the Casimir functions \( F_1 = (\gamma, \gamma) \), \( F_2 = (\mathbf{M}, \gamma) \). Making the change of variables \( \mathbf{M} \rightarrow \rho \mathbf{M}, \gamma \rightarrow \gamma \), we can express the bracket (6), the Hamiltonian (7), and the Casimir functions as

\[
\{M_i, M_j\} = \varepsilon_{ijk} \left( M_k - D \frac{\mathbf{M}}{\rho^2} a_k \gamma_k \right), \quad \{M_i, \gamma_j\} = \varepsilon_{ijk} \gamma_k, \quad \{\gamma_i, \gamma_j\} = 0,
\]

\[
H = \frac{1}{2} (1 - D(A\gamma, \gamma))(\mathbf{M}, A\mathbf{M}) + \frac{1}{2} D(A\mathbf{M}, \gamma)^2 + V(\gamma), \quad F_1 = (\gamma, \gamma), \quad F_2 = \rho(\mathbf{M}, \gamma).
\]

(8)

For a more detailed study of the structure of the nonlinear bracket (8), let us calculate the “monopole” [4], i.e., the integral \( \int \omega \) over the reduced two-dimensional sphere similar to the Poisson sphere, where \( \omega \) is the 2-form of the gyroscopic forces associated with the structure (8).

Indeed, the level of the Casimir functions

\[
(\gamma, \gamma) = 1, \quad \rho(\mathbf{M}, \gamma) = \rho
\]

(9)
can be transformed by

\[
\sigma = \mathbf{M} - \frac{\varepsilon}{\rho} \gamma
\]

(10)
to the cotangent bundle of the two-dimensional sphere

\[
(\gamma, \gamma) = 1, \quad (\sigma, \gamma) = 0.
\]

(11)

Let us parameterize the manifold (11) by the coordinates associated in the classical dynamics of solids with the Euler angles \( \theta, \varphi \) on the Poisson sphere and with the corresponding (for the structure \( c(3) \)) canonical momenta \( p_{\theta}, p_\varphi \):

\[
\gamma_1 = \sin \theta \sin \varphi, \quad \gamma_2 = \sin \theta \cos \varphi, \quad \gamma_3 = \cos \theta,
\]

\[
\sigma_1 = -\cot \theta \sin \varphi p_\varphi + \cos \varphi p_\theta, \quad \sigma_2 = -\cot \theta \cos \varphi p_\varphi - \sin \varphi p_\theta, \quad \sigma_3 = p_\varphi.
\]

(12)

Expressing the components \( \varphi, \theta, p_\varphi, p_\theta \) from (12) on the level (9) in terms of \( \mathbf{M}, \gamma \) and using the commutation relations (8), we obtain

\[
\{\theta, \varphi\} = \{p_\varphi, \theta\} = \{p_\theta, \varphi\} = 0, \quad \{p_\theta, \theta\} = \{p_\varphi, \varphi\} = 1, \quad \{p_\varphi, p_\theta\} = \frac{c \sin \theta}{\rho^3}.
\]

(13)
The “monopole” intensity is given by the integral

\[
Q = \int_{S^2} \frac{c \sin \theta}{\rho^3} d\theta d\varphi = \frac{2\pi c}{\sqrt{(1 - Da_1)(1 - Da_2)(1 - Da_3)}},
\]

(14)
i.e., just as in the classical dynamics of solids [4], the unremovable magnetic field \( (Q \neq 0) \) arises for a nonzero area constant.

For \( c = 0 \), the structure (8) reduces to the linear structure defined by the algebra \( c(3) \). For the Brunn potential \( V = (\varepsilon/2)(A\gamma, \gamma) \), the additional integral \( F_4 \) of Eqs. (1) can be expressed in terms of the new variables \( (\mathbf{M}, \gamma) \) as

\[
F_4 = (\mathbf{M}, \mathbf{M})(1 - D(\gamma, A\gamma)) - \frac{\varepsilon}{\det A}(\gamma, A\gamma),
\]

\[
A = \text{diag}(a_1, a_2, a_3),
\]

(15)
or

$$F_4 = (\mathbf{M}, \mathbf{M})(\gamma, \mathbf{A}^*\gamma) + \frac{\varepsilon D^2}{(1 - a_1^*)(1 - a_2^*)(1 - a_3^*)}(\gamma, \mathbf{A}^*\gamma),$$

$$\mathbf{A} = \text{diag}(a_1, a_2, a_3), \quad \mathbf{A}^* = \mathbf{E} - D \mathbf{A}.$$  \hspace{1cm} (16)

At the level $F_4 = c_4$, system (16) is equivalent the following one:

$$H^* = (\mathbf{M}, \mathbf{M}) + \frac{c}{(\gamma, \mathbf{A}^*\gamma)}, \quad c = \text{const}.$$  \hspace{1cm} (17)

This Hamiltonian occurred in the Braden paper [5] in connection with the study of potentials (admitting the separation of variables) on the two-dimensional sphere. More general results on rational potentials on the sphere which admit the separation of variables were given in [6].

Note that for system (6) with $V = 0$ the separating variables given in [5] coincide with those studied in [1], which makes it more natural to carry out transformations in a nonholonomic system, reducing them to the standard Hamilton–Jacobi method.

The question of whether Eqs. (1) are Hamilton equations without an additional time substitution, was posed in [2] and still remains unsolved. It is related to the problem of reducing a dynamical system with an integral invariant on the torus to a quasiperiodic flow [3].

REFERENCES


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