Dynamics of a Painlevé–Appell system

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A B S T R A C T

The dynamics of a Painlevé–Appell system consisting of two point masses joined by a weightless rigid rod is studied within two mechanical models, which describe different motion regimes. One of the masses can slide or can be supported at rest on a rough straight line. The boundaries of the region of definition of each of the models are presented, and the transitions between them are analysed for various friction coefficients.

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1. Introduction

The Painlevé–Appell problem of the dynamics of a pendulum whose fulcrum can move along a rough horizontal straight line will be investigated. We will assume that the contact with the supporting straight line is a point contact and that it obeys Coulomb’s law of dry friction

\[ F = -\mu NV/|V| \]

where \( F \) is the friction force, which counteracts sliding of the contact point with the velocity \( V \), \( \mu \) is the friction coefficient, and \( N \) is the absolute value of the normal reaction force of the supporting straight line.

Despite the simplicity of the formulation, at certain values of the friction coefficient (which are greater than a certain critical value) this problem exhibits paradoxes associated with the non-existence or non-uniqueness of the solution.

This system was proposed by Painlevé when he studied possible paradoxical solutions.\(^1\) A more detailed investigation was conducted by Appell,\(^2\) who obtained the equations of motion and performed an analysis of the system with small friction coefficients (\( \mu < 1 \), but the exact value of the critical friction coefficient was not determined). Several particular solutions were investigated in a more general formulation (in the presence of an additional external constraining force \( h \) besides the gravitational force).\(^3\) When \( h \to 0 \), a condition imposed on the friction coefficient, under which sudden stopping can occur as a result of an infinite increase in the friction force, was obtained.

The main results in Refs 2 and 3 are associated with arguments regarding the possibility or impossibility of motion when the contact point has a specific direction of motion. Not only particular solutions, but also the general results of a qualitative analysis of the dynamics, which will enable us below in the example of the classical Painlevé–Appell problem to draw conclusions regarding the behaviour of other systems with bilateral constraints, especially in the vicinity of values of the parameters at which paradoxical solutions are observed and, in addition, enable us to model the experiment and ensure observability of the effects described.

We note that there are many mechanical systems with Coulomb friction at a contact point and unilateral or bilateral constraints in which such paradoxes are encountered (for example, a body supported on a rough plane by one contact point,\(^4\) a brake shoe,\(^5\) a ladder resting on a horizontal floor and a vertical wall,\(^2,6\) a Painlevé–Klein system\(^7\) etc.). Despite the long history of the investigation of the systems indicated (beginning with Painlevé’s classical work\(^1\) and the subsequent discussion of the well-known mechanicians at the beginning of the twentieth century), an increase in interest in this subject was observed recently, especially in the investigation of solutions in the vicinity of paradoxical regions (for examples of modern analytical approaches using variational and asymptotic methods, see, e.g., monographs\(^8,9\) and the references therein).

Apart from the development of mathematical methods of investigation, there are two more principal reasons for the resumption of investigations of classical problems. First, computer simulation and visualization methods, which are needed, in particular, for the

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construction and analysis of phase portraits and regions of possible motion (RPMs), are undergoing rapid development. Analytical methods combined with the use of modern computer methods along with the development of bifurcation theory enable us to reveal qualitative features of the dynamics for various values of the parameters of the system.\textsuperscript{5} Second, the experimental facilities and methods for the high-speed recording of natural experiments and the data processing for observing different kinds of dynamic effects have been improved significantly. The main difficulty in setting up an experiment lies in establishing the initial conditions and parameters of the system that bring the system to a paradoxical solution (generally, the value of the critical friction coefficient for the problems just described is considerably greater than the values observed in nature). For this reason, new, more complex mechanical systems (for example, an inverted pendulum on a slider (IPOS),\textsuperscript{10} dynamic walkers\textsuperscript{11} etc.), whose theoretical analysis has revealed paradoxical solutions at values of the system parameters that are obtained in a natural experiment, are being developed. We also note that the opposite situation, in which so-called non-intuitive (non-obvious) motion is observed under easily implementable initial conditions, appears for several systems (with rolling).\textsuperscript{12–14}

Thus, to set up an experiment, it is first necessary to analyse the dynamics of the system as a whole and to model the behaviour of the system to obtain a more complete description of the theoretically predicted effects. As a rule, particular solutions alone do not suffice since theory and experiment cannot always provide the necessary quantitative agreement under the chosen initial conditions. Nevertheless, in a certain range of initial data it often suffices to obtain a qualitative agreement between experiment and theory.

Below, we will use modern computer capabilities and methods of qualitative analysis (that were developed for a system with a unilateral constraint\textsuperscript{6}) to investigate the dynamics of a Painlevé–Appell system and the special features of the solutions for friction coefficients that are smaller and greater than the critical value within two mechanical models: a pendulum and a sliding rod. The use of an illustrative method to construct the phase portraits has enabled us to consider the case when the friction coefficient is greater than the critical value even on the boundaries of the regions of possible motion when the motion mode changes.

2. Equations of motion and boundaries of regions of possible motion

A Painlevé–Appell system (for a schematic representation, see Fig. 1) is a planar pendulum whose fulcrum $M_1$ can slide along the rough straight line $Ox$ with the friction coefficient $\mu$. The pendulum is a rigid weightless rod of length $l$ with two point masses $M_1$ and $M_2$ of identical mass $m$ attached to its ends.

We introduce the following notation: $\theta$ is the angle $xM_1M_2$, $x_1$ is the abscissa of the point $M_1$, and $(x_2, y_2)$ denotes the coordinates of the point $M_2$. The following forces act on the point $M_1$: the weight $P = mg$, the reaction $R$ of the rod, and the normal component $N$ and the horizontal component $F$ of the reaction of the straight line $Ox$. The point $M_2$ is acted upon by the weight $P_2 = mg$ and the reaction $K$ of the rod.

If the force of the horizontal reaction $F$ along the directrix $Ox$ at the contact point does not exceed a certain limiting value (which depends on the friction coefficient), the system is set in motion in which the contact point does not slide, and the rod performs oscillation that can be described by the equations of a mathematical pendulum. If the reaction $F$ has the limiting value, the system slides at the contact point and can then perform oscillatory motions. Therefore (taking into account that the constraint is retaining at the point $M_1$), we will consider the motion of the system within two mechanical models: a) a pendulum ($x_1 = 0$), b) a sliding rod ($x_1 \neq 0$).

We will write all the equations below in dimensionless form. For this purpose, as the units of measure of length, time and force we choose the quantities $l, \sqrt{l/g}$ and $mg$.

**Pendulum.** In the pendulum model the contact point $M_1$ does not slide ($x_1 = 0$). In this case the configuration of the system is determined by the single generalized coordinate $\theta$, for which the equation of motion (the equation of a mathematical pendulum) has the form

$$\ddot{\theta} = \cos \theta$$

(2.1)

The region in which this model is applicable corresponds to the no-slip condition of point $M_1$, and the total reaction $F + N$ (see Fig. 1) from the straight line $Ox$ should lie within the cone of friction:

$$|\mu| |N| > |F|$$

(2.2)

To determine the regions corresponding to inequality (2.2), we write the equations of motion of points $M_1$ and $M_2$ in Cartesian coordinates:

$$\begin{align*}
\dot{x}_1 &= F + R \cos \theta, \quad \dot{y}_1 = 1 + N + R \sin \theta \\
\dot{x}_2 &= -R \cos \theta, \quad \dot{y}_2 = 1 - R \sin \theta
\end{align*}$$

(2.3)
In addition, we take into account that geometrical constraints of the form
\[ y_1 = 0, \quad x_2 = x_1 + \cos \theta, \quad y_2 = \sin \theta \tag{2.4} \]
are imposed on the coordinates of these points. Differentiating the constraints equalitie (2.4) and using Eqs (2.3), we obtain relations for the reactions
\[ N = -(1 + R \sin \theta), \quad F = -R \cos \theta, \quad R = \dot{\theta}^2 + \sin \theta \tag{2.5} \]

It follows from Eqs (2.3) that the cone of friction (2.2) or the region of possible motion in the pendulum model is defined by the relations (only the upper or only the lower plus and minus signs are taken simultaneously)
\[ (\mu \sin \theta \pm \cos \theta) \dot{\theta}^2 > -\mu (1 + \sin^2 \theta) \pm \sin \theta \cos \theta \tag{2.6} \]

In Fig. 2 the corresponding cones of friction (shaded regions) are presented in the (0, \( \dot{\theta} \)) planes for \( \mu = 0.3 \) (on the left) and \( \mu = 1 \) (on the right).

As a consequence of the symmetry about the \( \dot{\theta} = 0 \) axis, only the region \( \dot{\theta} > 0 \) is shown in Fig. 2 and below in Figs. 4, 5, 7, 8 and 9.

Sliding rod. In the sliding rod model, the contact point \( M_1 \) moves along the Ox axis with the velocity \( \dot{x}_1 \neq 0 \). This point is acted upon by the force of gravity \( F \), the reaction of the rod \( R \), the normal reaction \( N \) and the friction force \( F \) from the directrix \( Ox \), which is directed opposite to the velocity \( \dot{x}_1 \) and is defined, according to Coulomb’s law, by the expression
\[ F = -\varepsilon \mu |N| \quad \varepsilon = \text{sign} \dot{x}_1 \tag{2.7} \]

which should be substituted into equations of motion (2.3). The system of equations of motion obtained will be called System C.

When the rod moves, the reaction \( N \) can take both negative and positive values. To determine the reaction, we differentiate equality (2.4) twice and substitute the result into System C; we obtain the relation
\[ \alpha N + \beta |N| + \gamma = 0 \tag{2.8} \]

where
\[ \alpha = 1 + \cos^2 \theta, \quad \beta = \mu \varepsilon \sin \theta \cos \theta, \quad \gamma = 2 + \dot{\theta}^2 \sin \theta \tag{2.9} \]

which, in view of the equality
\[ \varepsilon |N| = \sigma N; \quad \sigma = \text{sign} \dot{x}_1 |N| \tag{2.10} \]

is more conveniently represented in the form
\[ N = -\gamma / f_\sigma, \quad f_\sigma = 1 + \cos^2 \theta + \sigma \mu \sin \theta \cos \theta \tag{2.11} \]

Since \( \gamma > 0 \) at any \( \theta \in [0, \pi] \), the sign of \( N \) is opposite to the sign of the function \( f_\sigma \), and, depending on \( \sigma, \mu \) and \( \theta \), can be either positive or negative. If a solution exists and is unique, the value of \( \sigma \) and the sign of \( N \) must match. Thus, for a specified friction coefficient \( \mu \), the region of possible motion in the motion model under consideration is determined by the values of \( \theta \) at which the matching condition holds.

To determine the regions where a solution exists for the normal reaction \( N \), we present different versions of the values of \( \sigma \) and the sign of \( N \) in the form of the following table:

<table>
<thead>
<tr>
<th>( \varepsilon )</th>
<th>( N &gt; 0 )</th>
<th>( N &lt; 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \sigma = 1, f_1 &lt; 0 )</td>
<td>( \sigma = -1, f_1 &gt; 0 )</td>
</tr>
<tr>
<td>-1</td>
<td>( \sigma = -1, f_1 &gt; 0 )</td>
<td>( \sigma = 1, f_1 &lt; 0 )</td>
</tr>
</tbody>
</table>

The function \( f_\sigma \) is positive-definite at any \( \theta \in [0, \pi] \) if
\[ \mu < (1 + \cos^2 \theta) / |\sin \theta \cos \theta| \tag{2.12} \]

and inequality (2.12) certainly holds if
\[ \mu < \mu^* = 2 \sqrt{2} \tag{2.13} \]
Remark. Inequality (2.12) was first obtained by De Sparre (see also Ref. 15, p. 91). He also showed that if this inequality is violated, an impact caused by the friction force occurs and the contact point to stop.

The table is used in the following manner. We first choose the required signs of the velocity \( \dot{x}_1 \) and the normal reaction \( N \) (a total of four different combinations), and we determine the corresponding values of \( \sigma = \text{sign}(\dot{x}_1, N) \). Then, depending on the value of \( \sigma \) obtained, we select the corresponding function \( f_{\sigma} \), bearing in mind that its sign is opposite to the sign of \( N \) chosen.

As a result, taking into account conditions (2.12) and (2.13), we obtain the following:

1) For \( \mu \leq \mu^* \), any direction of motion of the contact point \( M_1 \), only one of the functions \( f_1 \) and \( f_{-1} \) will satisfy the necessary condition from the table \( (f_{\pm 1} > 0) \); therefore, a solution of System \( C \) exists and is unique at any \( \theta \in [0, \pi] \); 2) for \( \mu > \mu^* \) there are regions of negative values of the functions \( f_1 \) and \( f_{-1} \) (Fig. 3); therefore, at certain values of \( \theta \) the necessary condition from the table can be satisfied either simultaneously by both functions (the solution of System \( C \) is non-unique) or by neither function (no solution of System \( C \) exists).

For example, when \( \mu \leq \mu^* \), we always have \( N < 0 \), but when the motion is to the right \( (\dot{x}_1 > 0) \), the solution requires that we set \( \sigma = -1 \), and when the motion is to the left \( (\dot{x}_1 < 0) \), we have to set \( \sigma = 1 \).

When \( \mu > \mu^* \) and the motion is to the left \( (\dot{x}_1 < 0) \), the solution is non-unique in the range \( \theta^*_1 < \theta < \theta^*_2 \) (\( \theta^*_1 \) and \( \theta^*_2 \) are roots of the function \( f_{-1} \), see Fig. 3), since in this range \( f_1 > 0 \) (which corresponds to the solution for \( N > 0 \)) and \( f_{-1} < 0 \) (which corresponds to the solution for \( N < 0 \)). Conversely, when the motion is to the right \( (\dot{x}_1 > 0) \) in the same range, neither \( f_1 \) nor \( f_{-1} \) satisfies the necessary conditions from the table \( (f_1 > 0, f_{-1} < 0) \). Thus, no solution of system \( C \) exists. In the range \( \theta^*_1 < \theta < \theta^*_2 \) (\( \theta^*_1 \) and \( \theta^*_2 \) are roots of the function \( f_1 \), see Fig. 3) the solution does not exist when \( \dot{x}_1 < 0 \) and is non-unique when \( \dot{x}_1 > 0 \).

Remark. The conditions for uniqueness of the solution and the normal reaction \( N \) in the system under consideration can be evaluated by analysing the extremum nature of the properties of the generalized acceleration \( \ddot{x}_1 \), which can be represented in the form of the left-hand side of equality (2.8) when notation (2.9) is used. It was shown that the normal reaction \( N \) is determined uniquely when \( |\beta| < \alpha \), which corresponds to condition (2.12). In addition, depending on the values of \( \mu \) and \( \theta \), two more cases are obtained: when \( \beta < -\alpha \), the stationary solution \( N = \gamma/(\alpha - \beta) \) exists, and when \( \beta > \alpha \), there are no stationary solutions.

Substituting expression (2.11) into System \( C \), we obtain equations which describe the dynamics of the system when the contact point \( M_1 \) slides:

\[
\begin{align*}
\dot{\theta} &= \frac{\dot{x}_2 (\cos \theta + \sigma \mu \sin \theta) + 2 \cos \theta + 2 \sigma \mu \sin \theta}{f_x}, \\
\dot{x}_1 &= \frac{\dot{x}_2 (\cos \theta + \sigma \mu \sin \theta) + \sin \theta \cos \theta + \sigma \mu (2 - \cos^2 \theta)}{f_x}
\end{align*}
\]

An important feature of Eqs (2.14) is the fact that their right-hand sides do not depend on the velocities \( \dot{x}_1 \) and \( \dot{x}_2 \) or the coordinates of points \( M_1 \) and \( M_2 \), but only on the value of \( \sigma \); therefore, by analogy with the problem of the fall of a thin rod (see Ref. 4), projections of the phase trajectories and of the boundaries of the regions of applicability onto the \( (\theta, \dot{\theta}) \) plane can be constructed. Since not only the sign of \( \dot{x}_1 \), but also the sign of \( N \) changes as the rod moves, there will be four such projections.

3. Dynamics of the system with various friction coefficients

Within the region of possible motion of each of the motion models used, the dynamics of the system is described by Eqs (2.5) or (2.14). Thus, specifying the initial data \( \theta(0), \dot{\theta}(0), \dot{x}_1(0) \) and numerically integrating the first equalities in (2.5) and (2.14), we obtain the phase trajectory in the space \( (\theta, \dot{\theta}, \dot{x}_1) \), whose projection onto the \((\theta, \dot{\theta})\) plane for a specified value of \( \sigma \) can be constructed correctly. When modern packages of analytical calculations are used, this problem is solved without notable difficulties. The main problem that arises during the analysis of the dynamics of systems with unilateral constraints that can be described within various mathematical models is the evaluation of the consistency and degree of smoothness of the phase trajectories when the common boundaries of the regions of possible motion intersect or abut.

In the system under consideration such a boundary is the \((\theta, \dot{\theta})(\dot{x}_1 = 0)\) plane, upon whose intersection the velocity of the contact point can change direction or remain equal to zero. Therefore, one of the problems considered below is the analysis of a phase trajectory upon the transition from motion with a fixed contact point to sliding and vice versa on the boundary of the cone of friction, as well as upon intersection of the \( \dot{x}_1 = 0 \) plane outside of the cone of friction.

A friction coefficient smaller than the critical value. It was shown above that when \( \mu \leq \mu^* \), the normal reaction \( N < 0 \) for any \( \theta \in [0, \pi] \); therefore, it is sufficient to construct two projections of the phase trajectory onto the \( \dot{x}_1 = 0 \) plane, which correspond to \( \sigma = 1 \) when \( \dot{x}_1 > 0 \) and to \( \sigma = -1 \) when \( \dot{x}_1 < 0 \). Phase trajectories of a pendulum are shown in the left-hand part of Fig. 4, and projections of the phase trajectories of a sliding rod onto the \((\theta, \dot{\theta})\) plane for \( \sigma = -1 \) and \( \sigma = 1 \) when \( \mu < \mu^* \) (\( \mu = 0.3 \)) are shown in the middle and right-hand parts of the figure.
The cone of friction of the pendulum is shaded in all the parts of the figure. It is seen that the phase trajectories adjoin consistently on the boundary of the cone of friction, i.e., when this boundary is intersected, the direction of motion along the phase trajectory remains unchanged.

To determine the direction of motion of the contact point at the instant of time when boundary of the cone of friction is intersected, the direction of the acceleration \( \dot{x}_1 \) in the vicinity of the \( \dot{x}_1 = 0 \) plane must be determined from the first equation of (2.14).

In Fig. 5 the regions in which the acceleration \( \dot{x}_1 \) is directed toward the \( \dot{x}_1 = 0 \) plane when \( \sigma = \pm 1 \) and \( \mu < \mu^* \) (\( \mu = 0.3 \)) are shaded. It is seen that the acceleration \( \dot{x}_1 \) is directed toward the \( \dot{x}_1 = 0 \) plane everywhere in the vicinity of the cone of friction.

Therefore, when \( \mu \leq \mu^* \), motion with a fixed contact point is stable in the cone of friction.

Thus, when the pendulum intersects the \((\theta, \dot{\theta})\) plane in the cone of friction, the contact point remains fixed until the phase trajectory intersects the boundary of the cone of friction (2.6) again.

In addition, the direction in which the cone of friction begins when the boundary of the cone of friction (2.6) is intersected can be determined from Fig. 5: to the left when \( \theta > \pi/2 \) (\( \dot{x}_1 < 0 \), \( \sigma = 1 \)) and to the right when \( \theta < \pi/2 \) (\( \dot{x}_1 > 0 \), \( \sigma = -1 \)), since the acceleration \( \dot{x}_1 \) is directed away from the \((\theta, \dot{\theta})\) plane when these are the signs of the velocity of the contact point.

We also note that the regions of uncertainty of the direction of sliding are absent (when the acceleration of the contact point can be directed simultaneously in both directions away from the plane). Thus, when the phase trajectory intersects the \( \dot{x}_1 = 0 \) plane, reversal of the direction of motion of the contact point occurs (the sign of \( \sigma \) changes) outside of the cone of friction.

Thus, when \( \mu \leq \mu^* \), the transition between the different motion modes is consistent.

We will demonstrate this in an example. The upper part of Fig. 6 presents the projection of the phase trajectory obtained when

\[
\mu = 0.3, \quad \theta(0) = 3\pi/4, \quad \dot{\theta}(0) = 0, \quad \dot{x}_1(0) = 0.3
\]

The cone of friction is also shown (the shaded areas). The parts of the phase trajectory lying in the \( \dot{x}_1 = 0 \) plane in cone of friction (2.6) are denoted by dashed lines. The points \( c_n \) correspond to the instants of time \( t_n \) \((n = 0, 1, ..., 7)\), which are listed to the right of the upper part of Fig. 6.

The phase trajectory obtained, which consists of several parts with discontinuities at the points \( c_n \), is illustrated by the corresponding plots of the dependence of the velocity \( \dot{x}_1 \) of the point \( M_1 \) (the middle part of Fig. 6) and of the projections of the velocity \( \dot{x}_2 \) and \( \dot{y}_2 \) of the point \( M_2 \) (the lower part) on the time, which were obtained under conditions (3.1).

The velocity decreases from the initial value \( \dot{x}_1(0) \) at the point \( c_0 \) and becomes equal to zero at the instant of time \( t_1 = 0.47 \) (the point \( c_1 \)), and since this point lies outside of the cone of friction, the direction of motion changes (\( \sigma \) also changes sign). Next, the velocity of the point \( M_1 \) becomes equal to zero at \( t_2 = 0.91 \) at the point \( c_2 \), which lies in the cone of friction; therefore, the motion is now described by system (2.3), and the velocity is \( \dot{x}_1 = 0 \) until boundary (2.6) is intersected at the point \( c_3 \) when \( t_3 = 2.02 \). Then motion occurs with a positive velocity and is described by system (2.4) and so on.

The amplitude of the oscillations of the angle \( \theta \) upon further alternating of such motions constantly decreases. This ultimately results in the fact that the phase trajectory will lie completely within the cone of friction. Also, since the motion is now described by Eqs (2.5) (the equations of a mathematical pendulum), the trajectories become periodic.

The larger the value of the friction coefficient \( \mu \), the larger the region in the \((\theta, \dot{\theta})\) plane that is occupied by the cone of friction when \( \dot{x}_1 = 0 \) (see Fig. 2). Thus, achievement of a periodic phase trajectory can occur already upon the first intersection of the \( \dot{x}_1 = 0 \) plane.

**A friction coefficient greater than the critical value.** We recall that when \( \mu > \mu^* \), regions of negative values of the functions \( f_\sigma \) appear (see Fig. 3). If the initial conditions for the angle \( \theta \) are chosen from these regions, the conditions of the table in Section 2 will be satisfied by either two values of \( \sigma \) or by no value.
To analyse the dynamics of system (2.14), we construct projections of the phase trajectory onto the \((\theta, \dot{\theta})\) plane. We note that these regions are defined only by the roots of the functions \(f_\sigma\) and do not depend on \(\dot{\theta}\).

The projections for all four combinations of the signs of \(x_1\) and \(\sigma\) must be considered; for each of them we will first establish the condition which determines the region of possible motion for the sliding rod model (see Figs. 7a, 7b, 7c and 7d):

a) for \(x_1 > 0, \sigma = -1\), it is the condition \(N < 0\), thus, \(f_\sigma > 0\), or \(\theta \in (0, \theta_1) \cup (\theta_2, \pi)\),

b) for \(x_1 < 0, \sigma = 1\), it is the condition \(N < 0\), thus, \(f_\sigma > 0\), or \(\theta \in (0, \theta_1) \cup (\theta_2, \pi)\),

c) for \(x_1 > 0, \sigma = 1\), it is the condition \(N > 0\), thus, \(f_\sigma < 0\), or \(\theta \in (0, \theta_1) \cup (\theta_2, \pi)\),

d) for \(x_1 < 0, \sigma = -1\), it is the condition \(N > 0\), thus, \(f_\sigma < 0\), or \(\theta \in (0, \theta_1) \cup (\theta_2, \pi)\).

When two projections are superimposed, for example, when \(x_1 > 0\) and \(\sigma = \pm 1\) (Figs. 7a and 7c), we find that two different phase trajectories correspond to the same initial data from the region \(\theta \in (\theta_1, \theta_2)\): one intersects the boundaries of the region indicated, and the
other asymptotically tends to $\theta_1^*$. Similarly, when the two projections for $\dot{x}_1 < 0$ and $\sigma = \pm 1$ (Figs. 7b and 7d), which correspond to different directions of the reaction force $N$, are superimposed, we find that two different phase trajectories correspond to the same initial data from the region $\theta \in (\theta_1^*, \theta_2^*)$: one of them intersects the boundaries of the region indicated, and the other asymptotically tends to $\theta_1^*$.

Nevertheless, these solutions are obtained for different directions of the reaction force $N$, and without external perturbations the transition from one solution to the other is impossible.

Thus, when the direction of the reaction force is specified at the initial instant, a single solution exists for any values of $\theta$ and $\dot{\theta}$ from the region of possible motion.

If the velocity of the contact point becomes equal to zero as the system moves, then, as was noted above, it can change direction or remain equal to zero. To determine the possibility of a particular motion, it is necessary, as in the case $\mu < \mu^*$, to construct regions in which the acceleration of the contact point near $\dot{x}_1 = 0$ is directed toward the $(\theta, \dot{\theta})$ plane (they are shaded in Fig. 8) for $\sigma = \pm 1$ and $\mu > \mu^*$ ($\mu = 3$).

It is seen that within the cone of friction, including the regions of non-uniqueness, the acceleration of the contact point is directed everywhere toward the plane. The regions where no solution exists are exceptions, but by beginning the motion from the region of possible motion of a system with sliding at the contact point, it is impossible reach this region (see Fig. 7).

Thus, when $\mu > \mu^*$, the motion with a fixed contact point is stable in the cone of friction. When the phase trajectories intersect the $(\theta, \dot{\theta})$ plane, further motion of the system will occur at a zero value of the velocity of the contact point until the boundary of the cone of friction is intersected.

Projections of phase trajectories (2.3) for $\dot{x}_1 = 0$ are presented in Fig. 9. The cone of friction corresponding to $\mu = \pm 1$ is shaded. When the phase trajectories intersect the boundary of the cone of friction (which is denoted by the heavy line), as in the case $\mu < \mu^*$, sliding of the contact point to the left ($\dot{x}_1 < 0$, $\sigma = 1$) at $\theta > \pi/2$ up to $\theta = \pi$ or to the right ($\dot{x}_1 > 0$, $\sigma = -1$) at $\theta < \pi/2$ up to $\theta = 0$ begins consistently and continuously, according to Fig. 8.

**Value of the critical friction coefficient when $m_1 \neq m_2$.** The motion of a Painlevé–Appell system with equal masses of the points $M_1$ and $M_2$ was considered above. In that case the corresponding critical value of the friction coefficient $= 2\sqrt{2}$ is practically unattainable in a natural experiment. On the other hand, all the qualitative features of the dynamics of the system obtained when $\mu > \mu^*$ will exist even if the value of $\mu^*$ can somehow be decreased. One of the possible versions is to change the relation between the masses $m_1$ and $m_2$.
Thus, we will consider a Painlevé–Appell system with different masses $m_1$ and $m_2$ of the material points $M_1$ and $M_2$, which are joined by a weightless rod of length $l$. Then in the equations of motion of the rod (system C) the multiplier $A = m_2/m_1$ appears in front of $\dot{x}_2$ and $\dot{y}_2$, and by analogy with the preceding, we obtain

$$N = \frac{A \cos^2 \theta + \sin^2 \theta + A \dot{\theta}^2 \sin \theta \cos \theta}{1 + A \cos^2 \theta + A \mu \sin \theta \cos \theta}$$

(3.2)

The numerator in this expression is positive-definite for any $\theta \in [0, \pi]$. Therefore, the single-valuedness of the reaction $N$ and the uniqueness of the solution depend on the compatibility of the sign of the denominator in (3.2) and the value of $\sigma$. Analysing the denominator, we find that it is everywhere positive, in agreement with the uniqueness of the solution ($N < 0$) for any $\theta \in [0, \pi]$, if

$$\mu < \mu^* = 2\sqrt{1 + A/A}$$

(3.3)

From the graph of the dependence of $\mu^*(A)$ (3.3) presented in Fig. 10, it is seen that the value of the critical friction coefficient obtained in a natural experiment ($\mu^* < 1$) corresponds to $A > 5$ ($m_2 > 5m_1$).

At large values of $A$ ($m_2 \gg m_1$) the coefficient of friction $\mu^*$ tends to zero. Therefore, in this case we can neglect the mass and friction at the point $M_1$, i.e., the limiting case of the system is a weightless rod, to one end of which a weight is attached, while the other end moves along a smooth directrix (see Ref. 17, Chapter 1, Section 5).

4. Conclusion

A visual method for constructing phase portraits was used to investigate a Painlevé–Appell system, for which it is known that there is a critical value of the friction coefficient $\mu_c = 2\sqrt{2}$ when the masses of the weights are equal. When it is exceeded, the region of possible values is restructured significantly as the contact point slides, and, in addition, regions of non-existence and non-uniqueness of the solution appear.

It was shown that for friction coefficients $\mu < \mu^*$ the solution exists everywhere and is unique, that the pendulum and sliding rod models correctly describe the possible motions, and that the transition between different motion modes (with and without sliding of the contact point) occurs consistently. It was shown in a specific example that motion with a non-zero velocity of the contact point in a finite time reduces to ordinary oscillations of the pendulum at a fixed contact point.

For $\mu > \mu^*$ the boundaries of the region of possible motions and the phase trajectories corresponding to the different combinations of the signs of the reaction $N$ and the velocity of the contact point $\dot{x}_2$ were constructed. It was found that in the region of non-uniqueness of the solution for the reaction $N$ there are different phase trajectories that correspond to the same initial conditions. However, for the direction of the reaction determined at the initial instant of time, a single solution exists for any values of $\dot{\theta}$ from the region of possible motion, and a transition between the solutions is impossible without external perturbations. A dynamical analysis showed that trajectories from the region of uniqueness never pass into the region of paradoxes.

In addition, it was shown that when $\mu > \mu^*$, a stable solution with a fixed contact point exists for any initial conditions from the region of possible motion of the motion models considered and that it leads to final oscillatory motion similarly to the case of a small friction coefficient. If the phase trajectory corresponding to the pendulum model reaches the boundary of the cone of friction, sliding of the contact point in a direction that depends on the position of the point on this boundary consistently begins.

It is known that systems of a similar kind with bilateral constraints cannot be described only within the model of an absolutely rigid body; however, the existing theoretical approaches are extremely cumbersome and were not previously subjected to experimental verification. The dependence obtained above of the critical value of the friction coefficient on the relation between the masses of the weights (3.3) enables us to attain the critical values of the friction coefficient obtained in practice and points to the possible paths for experimental investigations.
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