On the Mechanism of Stability Loss

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Received January 12, 2009

Dedicated to Academician Viktor Antonovich Sadovnichii
on the occasion of his seventieth birthday

Abstract—We consider linear systems of differential equations admitting functions in the form of quadratic forms that do not increase along trajectories in the course of time. We find new relations between the inertia indices of these forms and the instability degrees of the equilibria. These assertions generalize well-known results in the oscillation theory of linear systems with dissipation and clarify the mechanism of stability loss, whereby nonincreasing quadratic forms lose the property of minimum.

DOI: 10.1134/S0012266109040041

1. STABILITY LOSS IN GYROSCOPIC SYSTEMS WITH ENERGY DISSIPATION

The equations of motion of a system with gyroscopic forces and energy dissipation linearized in a neighborhood of the equilibrium $x = 0$ have the form

$$K \ddot{x} + (\Gamma + D) \dot{x} + P x = 0, \quad x \in \mathbb{R}^k. \quad (1.1)$$

Here $K$ is the operator of inertia (a positive definite symmetric matrix), $\Gamma$ is the skew-symmetric matrix of gyroscopic forces, the symmetric matrix $P$ defines the potential energy

$$V(x) = \frac{1}{2}(P x, x)$$

of the system, and the positive semidefinite symmetric matrix $D$ defines the Rayleigh dissipation function

$$\Phi(x) = \frac{1}{2}(D \dot{x}, \dot{x}) \geq 0.$$

Equation (1.1) implies the total energy evolution law

$$(T + V)' = -2\Phi, \quad (1.2)$$

where

$$T = \frac{1}{2}(K \dot{x}, \dot{x})$$

is the kinetic energy of the system.

We assume that $|P| \neq 0$. Then $x = 0$ is an isolated equilibrium of system (1.1). Its Poincaré instability degree $\iota^-$ is the nonnegative inertia index of the quadratic form $V$. Since the quadratic form $T$ is positive definite, it follows that $\iota^-$ coincides with the inertia index of the total energy $T + V$.

The *instability degree* $u$ of the equilibrium $x = 0$, $\dot{x} = 0$ is defined as the number of roots of the characteristic equation

$$|\lambda^2 K + (\Gamma + D)\lambda + P| = 0 \quad (1.3)$$
in the right complex half-plane (with regard of multiplicities). By the classical Kelvin theorem,

$$u \equiv i^- \pmod{2} .$$

(1.4)

On the other hand, as was shown in [1], if $D > 0$ (the case of full dissipation), then $u = i^-$. It was shown in [2] that

$$u \leq i^-$$

(1.5)

for systems with partial energy dissipation ($D \geq 0$). In fact, this inequality remains valid in the more general case in which the kinetic energy $T$ is nondegenerate but not necessarily positive definite. Then $i^-$ in (1.4) has the meaning of the negative inertia index of the quadratic form $T + V$. A formula for the difference $i^- - u$ in terms of the quadratic matrix pencil $\lambda^2 T + \lambda (\Gamma + D) + P$ was obtained in [3].

Relations (1.4) and (1.5) imply the following simple but important assertion.

**Theorem 1.** If $i^- = 1$, then the characteristic polynomial (1.3) has a unique real root in the right complex half-plane.

Indeed, in this case, we have $u = 1$, and the unique root is necessarily real.

Now suppose that the system continuously depends on a parameter $\varepsilon$ and the potential energy takes the minimum value at the point $x = 0$ for $\varepsilon < 0$, becomes degenerate for $\varepsilon = 0$ ($|P| = 0$), and loses the property of minimum for $\varepsilon > 0$, its inertia indices $i^-$ and $i^+$ being equal to 1 and $k - 1$, respectively. Thus, if $\varepsilon < 0$, then the equilibrium $x = 0$ is stable and the characteristic polynomial (1.3) has no roots in the right complex half-plane. When passing through the bifurcation point $\varepsilon = 0$, there appears exactly one positive real root, and the remaining roots are pure imaginary or lie in the left half-plane. Needless to say, this real root continuously depends on $\varepsilon$ and tends to zero (remaining positive) as $\varepsilon \to +0$. This observation is just the mechanism of stability loss.

Our aim is to generalize this result to systems of most general form.

**2. LINEAR AUTONOMOUS SYSTEMS**

Consider the dynamical system

$$\dot{x} = Ax + \cdots, \quad x \in \mathbb{R}^n,$$

(2.1)

where dots stand for terms of order $\geq 2$. We assume that $A$ is a nondegenerate operator ($|A| \neq 0$); in particular, $x = 0$ is an isolated equilibrium.

Suppose that system (2.1) admits a function $F : \mathbb{R}^n \to \mathbb{R}$ such that

$$F(x) = \frac{1}{2} (Bx, x) + \cdots, \quad B^T = B,$$

(2.2)

and $\dot{F} \leq 0$. The quadratic form in this expansion is assumed to be nondegenerate ($|B| \neq 0$). Obviously, the derivative of the quadratic form in (2.2) according to the linearized system $\dot{x} = Ax$ is also nonnegative. It is this fact that we use in forthcoming considerations.

The second-order system (1.1) can be represented in the form (2.1). In this case, we have $n = 2k$, and the function $F$ has the meaning of the total energy.

Let $i^\pm$ be the inertia indices of the Morse function $F$ at the critical point $x = 0$, and let $u$ be the instability index of system (2.1), i.e., the number of eigenvalues of the operator $A$ in the right complex half-plane. The congruence (1.4) was proved in [4] in this most general case.

**Theorem 2.** One has $u \leq i^-$. This theorem generalizes the result in [2] to dynamic equations with dissipation and even admits a simpler proof.

**Remark 1.** Theorem 2 remains valid in the lack of dissipation. Then $F$ is a first integral of system (2.1). A stronger version of Theorem 2 in the conservative case was proved in [5].
Proof of Theorem 2. Let us replace the linearized system (2.1) by the system
\[ \dot{x} = Ax - \mu Bx, \]  
where \( \mu \) is a nonnegative real parameter. Obviously,
\[ \frac{1}{2}(Bx, x) \leq -\mu(Bx, Bx). \]  
Consequently, if \( \mu > 0 \), then this derivative is negative definite. Then, by the Ostrowski–Schneider theorem [6], the number of eigenvalues of the operator \( A - \mu B \) in the right complex half-plane is equal to \( i^- \). The remaining eigenvalues can lie in the left half-plane or on the imaginary axis. Let us show that the latter possibility is not realized. If it is, then the linear system (2.3) has a nontrivial \( \tau \)-periodic solution \( t \mapsto \hat{x}(t) \). Let us substitute this solution into (2.4) and average both sides over the interval \([0, \tau]\):
\[ 0 = \frac{1}{\tau} \int_0^\tau (B\hat{x}, \dot{\hat{x}}) \, dt \leq -\frac{\mu}{\tau} \int_0^\tau (B\hat{x}, B\hat{x}) \, dt. \]
Since \( B \) is nondegenerate and \( \hat{x} \neq 0 \), it follows that the integral on the right-hand side is positive. We have arrived at a contradiction.

Now we let the parameter \( \mu \) tend to zero. The eigenvalues of the operator \( A - \mu B \) are continuous with respect to \( \mu \). More precisely, they can be numbered so as to ensure that they continuously depend on \( \mu \). Note in passing that the dependence on the parameter \( \mu \) need not be smooth in the case of multiple eigenvalues. But continuity is sufficient for our purpose. Since the eigenvalues of the operator \( A - \mu B \) do not lie on the imaginary axis for all sufficiently small \( \mu > 0 \), it follows that, for \( \mu = 0 \), the number of eigenvalues to the left and right of the imaginary axis can only diminish: part of them can reach the imaginary axis for \( \mu = 0 \). The proof of the theorem is complete.

The congruence (1.4), which holds in the general case, together with Theorem 2, implies the following assertion.

Corollary 1. Let \( i^- = 1 \). Then only one eigenvalue of the operator \( A \) lies in the right complex half-plane, and it is a real positive number.

This assertion clarifies the mechanism of stability loss for equilibria of dissipative systems of general form.

As an illustrative example, consider the second-order linear system
\[ \ddot{x} = kx + Sx, \quad x \in \mathbb{R}^n. \]  
Here \( k \) is a real number (so that the first term on the right-hand side represents a central force), and \( S \) is a nondegenerate skew-symmetric matrix. (In particular, \( n \) is even.) The second term on the right-hand side in (2.5) is an additional positional force applied to the system. Since \( S \) is skew-symmetric and \( S \neq 0 \), it follows that this force cannot be potential. Set
\[ F = (S\dot{x}, x). \]  
The function \( F \) plays the role of angular momentum of a many-dimensional dynamical system. Then
\[ \dot{F} = -(Sx, Sx) \leq 0. \]  
Since \( |S| \neq 0 \), it follows that the quadratic form (2.6) in the variables \( x \) and \( \dot{x} \) is nondegenerate and its inertia index is equal to \( n \). Consequently, \( u \leq n \) and \( u \) is even. By rewriting relation (2.7) for the function \(-F\) and by taking into account the nondegeneracy of the matrix \( S \), from the Barbashin–Krasovskii theorem, we derive the instability of the equilibrium \( x = 0 \) (regardless of the sign of the elasticity constant \( k \)). Consequently, \( u \geq 1 \). Since \( u \) is even, we have \( u \geq 2 \). In particular, for \( n = 2 \), the instability degree of the linear system (2.5) is equal to 2.
3. LINEAR SYSTEMS WITH PERIODIC COEFFICIENTS

Under some additional conditions, the results in Section 2 can be extended to the more general linear systems

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \quad (3.1)$$

where the operator $A$ is $\tau$-periodic in time. Let

$$F(x, t) = \frac{1}{2}(B(t)x, x) \quad (3.2)$$

be a quadratic form with a $\tau$-periodic symmetric operator $B$ such that its derivative $\dot{F}$ according to system (3.1) is nonpositive. Obviously, $\dot{F}$ is a quadratic form in $x$ with periodic coefficients.

The straightforward way is to use the Floquet–Lyapunov theorem, which claims that there exists a linear $\tau$-periodic change of variables

$$x \mapsto C(t)x \quad (3.3)$$

reducing system (3.1) to autonomous form. More precisely, such a change of variables is only $2\tau$-periodic in the general case; however, this is inessential for forthcoming considerations. In the new variables, the quadratic form (3.2) is equal to

$$\frac{1}{2}(\hat{B}x, x), \quad \hat{B} = C^TBC \quad (3.4)$$

and Eq. (3.1) acquires the form

$$\dot{x} = \hat{A}x, \quad \hat{A} = C^{-1}(AC - \dot{C}) = \text{const.} \quad (3.5)$$

Obviously, the derivative of the form (3.4) according to system (3.5) is also nonpositive. Moreover, if $\hat{F} = 0$, then the periodic quadratic form (3.4) is a first integral of system (3.5).

In the general case, the form (3.4) explicitly depends on time, and this fact complicates further investigation. We expand the matrix $\hat{B}$ in a Fourier series in time,

$$\hat{B}(t) = \sum_{m=-\infty}^{\infty} B_m \exp \left( \frac{i \times 2\pi mt}{\tau} \right),$$

where the coefficients $B_m$ are symmetric $n \times n$ matrices. Then the derivative of (3.4) according to system (3.5) acquires the form

$$\sum \left( \left[ \frac{i \pi m}{\tau} B_m + B_m \hat{A} + \hat{A}^T B_m \right] x, x \right) \exp \left( \frac{i \times 2\pi mt}{\tau} \right).$$

Since this form is nonpositive for all $t$, we see that its average over the period is nonpositive as well. Consequently,

$$((B_0 \hat{A} + \hat{A}^T B_0)x, x) \leq 0. \quad (3.6)$$

Set

$$F_0 = \frac{1}{2}(B_0x, x) \quad (3.7)$$

Then $\dot{F}_0 \leq 0$. To use Theorem 2, one should verify the nondegeneracy of the operator $\hat{A}$ and the quadratic form (3.7).

Let $\rho_1, \ldots, \rho_n \in \mathbb{C}$ be the multipliers of the periodic linear system (3.1). The condition $|\hat{A}| \neq 0$ is equivalent to the assumption that $\rho = 1$ is not one of the multipliers. In fact, the period of the transformation (3.3) is equal to $2\tau$; therefore, the monodromy operator of system (3.1) should be squared. Consequently, the nondegeneracy condition is equivalent to the condition $\rho \neq \pm 1$. 

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The problem on the nondegeneracy of the quadratic form (3.6) is more complicated. The following example provides some notion of the related difficulties. Let

\[ B(t) = C^T(t)DC(t), \]

where

\[ D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad C(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}. \]

For each \( t \), the symmetric matrix \( B(t) \) is nondegenerate; moreover, the inertia index of the corresponding quadratic form is the same \( (i^- = 1) \); however,

\[ \int_0^{2\pi} B(t) \, dt = 0. \]

Here we solve the problem under the assumption that the derivative \( \dot{F} \) according to system (3.1) is negative definite for almost all \( t \). Then so is the derivative of the \( t \)-periodic quadratic form (3.4) according to system (3.5). In particular, the derivative \( \dot{F}_0 \) is negative definite. But

\[ \dot{F}_0 = (\hat{A}x, B_0x). \] (3.8)

If the quadratic form (3.7) is degenerate, then \( B_0x = 0 \) for some \( x \neq 0 \). Then, by formula (3.8), we obtain the relation \( \dot{F}_0 = 0 \) at that point, which contradicts the negative definiteness of the form \( \dot{F}_0 \).

Let \( i^- \) be the negative inertia index of the quadratic form (3.7), and let the instability degree \( u \) of the zero solution of the linear system (3.1) be defined as the number of multipliers lying outside the unit disk in \( \mathbb{C} \). The Ostrowski–Schneider theorem implies the following assertion.

**Theorem 3.** If \( \dot{F} \) is negative definite for almost all \( t \) and the linear system (3.1) is nondegenerate, then \( u = i^- \).

It remains to consider the meaningful problem on the computation of the inertia index \( i^- \). One should not expect that it suffices to perform the averaging of the original quadratic form with respect to time: first one should pass to new variables with the use of the linear change of variables (3.3), which is not given a priori. The case in which \( F \) is a first integral of the linear system (3.1) is especially interesting and especially complicated.

Theorem 3 can be extended to arbitrary Lyapunov reducible nonautonomous linear systems (3.1) that admit a quadratic form (3.2) with bounded coefficients that is nonincreasing along the solutions. We assume that all entries of the matrix \( A \) are bounded functions of time and that the linear transformation reducing system (3.1) to the linear system (3.5) with constant coefficients is a Lyapunov transformation. Recall that the spectrum of the linear system (3.1) is the set of Lyapunov characteristic exponents of all of its solutions. The spectrum is preserved under Lyapunov transformations. The spectrum contains at most \( n \) real numbers. The complete spectrum of system (3.1) takes into account the multiplicities of the characteristic exponents and consists of exactly \( n \) real numbers.

Thus, let the linear system (3.1) be reduced to the autonomous system (3.5), and let the quadratic form (3.2) be reduced to the form (3.4) with bounded coefficients. We suppose that the matrix \( \hat{B}(t) \) is Cesaro integrable; namely, there exists a limit

\[ \lim_{s \to \infty} \frac{1}{s} \int_0^s \hat{B}(t) \, dt = B_0. \] (3.9)

In particular, in the periodic case, this limit always exists and is equal to the average over the period. Note also that if \( \hat{B}(t) \to B_0 \) as \( t \to \infty \) in the ordinary sense, then relation (3.9) holds. The following assertion is a generalization of Theorem 3.
Theorem 4. Suppose that $\tilde{F}$ is negative definite for almost all $t$ and the spectrum of the linear system (3.1) does not contain zero. Then the number of positive elements in the complete spectrum of system (3.1) coincides with the negative index of inertia of the quadratic form $(B_0 x, x)$.

Proof. The proof of this theorem is based on the same ideas and the simple remark that the complete spectrum of the linear system (3.5) consists of the real parts of characteristic numbers of the constant matrix $\hat{A}$. Since the entries of the matrix $\hat{B}(t)$ are bounded functions of time, it follows that the Cesaro limit of $(\hat{B}(t))_t$ is zero, and we again obtain relation (3.6).

Remark 2. The Cesaro convergence (3.9) can be replaced by any summation method that includes the Cesaro method. For example, let a positive function $t \mapsto \lambda(t)$ be nonincreasing, and let

$$\int_0^\infty \lambda(t) \, dt = \infty.$$ 

Then condition (3.9) can be replaced by the condition

$$\lim_{s \to \infty} \frac{\int_0^s \hat{B}(t) \lambda(t) \, dt}{\int_0^s \lambda(t) \, dt} = B_0.$$  

(3.10)

If $\lambda(t) = \text{const} > 0$, then we obtain relation (3.9). But if $\lambda(t) = (t + 1)^{-\alpha}$, $0 < \alpha \leq 1$, then condition (3.10) is weaker. Continuous summation methods were discussed, e.g., in [7].

4. DISCRETE CASE

The study of linear periodic systems can be performed in a different way by means of the analysis of a mapping

$$x \mapsto \Omega x, \quad \Omega = \text{const},$$  

(4.1)

and a quadratic form

$$F(x) = (B x, x)$$  

(4.2)

such that $F(\Omega x) \leq F(x)$ for all $x \in \mathbb{R}^n$. The latter implies the following:

$$\Omega^T B \Omega - B = S,$$  

(4.3)

where $S$ is a nonpositive symmetric operator $[(S x, x) \leq 0 \text{ for all } x \in \mathbb{R}^n]$. If $S = 0$, then formula (4.2) defines a first integral of the linear mapping (4.1).

The linear operator $\Omega$ can be treated as the monodromy operator for the linear system (3.1), and the quadratic form (4.2) is the function (3.2) at a fixed time. If $B(t)$ is a nondegenerate matrix for all $t$, then, by using a linear change of variables periodic in $t$, one can reduce the form (3.2) to a constant quadratic form; this will be the quadratic form (4.2). Conversely, a system of differential equations has been reduced in Section 3 to an autonomous form. The eigenvalues of the operator $\Omega$ coincide with the multipliers of the original linear system.

We assume that the following two nondegeneracy conditions are satisfied:

(i) the numbers $\rho = \pm 1$ are not eigenvalues of the operator $\Omega$;
(ii) $|B| \neq 0$.

We again assume that $i^-$ is the negative inertia index of the nondegenerate quadratic form (4.2) and $u$ is the instability degree of the mapping (4.1), which is defined as the number of eigenvalues lying outside the unit disk on the complex plane.

Theorem 5. One has the congruence

$$u \equiv i^- \pmod{2}.$$  

(4.4)

Corollary 2. If $i^-$ is odd, then the fixed point $x = 0$ of the mapping (4.1) is exponentially unstable.
If \( i^- = 0 \) [then \( x = 0 \) is the point of minimum of the function \( (4.2) \)], then the fixed point \( x = 0 \) is stable. This is a discrete analog of the Lyapunov theorem. Suppose that, as a parameter varies, the quadratic form loses the property of minimum and one minus \((i^- = 1)\) appears in its canonical representation (where \( B \) is a diagonal matrix). Then the fixed point \( x = 0 \) loses stability and becomes exponentially unstable. This remark describes the mechanism of stability loss for the case in which the “energy” loses the property of minimum.

**Proof of Theorem 5.** First, note that it suffices to prove the congruence \((4.4)\) for the number of real multipliers, since the complex multipliers lying outside (inside) the unit disk on \( \mathbb{C} \) occur in pairs.

We use the operator identity

\[
(\Omega^T - E)B(\Omega + E) = \Gamma + S, \quad \Gamma = \Omega^TB - B\Omega. \tag{4.5}
\]

In its derivation, we have used relation \((4.3)\). Obviously, \( \Gamma \) is a skew-symmetric operator. Let

\[
f(g) = |\Omega - \lambda E| = |\Omega^T - \lambda E|
\]

be the characteristic polynomial. It follows from \((4.5)\) that

\[
f(1)f(-1)|B| = (-1)^n| - S - \Gamma|. \tag{4.6}
\]

Since \( f(\pm 1) \neq 0 \) and \( |B| \neq 0 \), it follows that \( -S - \Gamma \) is a nondegenerate matrix as well. Further, since \( -S \) is symmetric and positive semidefinite, we have \( | - S - \Gamma | > 0 \) (Lemma 2 in [4]).

Consequently, by \((4.6)\),

\[
\text{sgn} f(1)f(-1) = (-1)^{n-i}. \tag{4.7}
\]

First, let \( n \) be even. If the index \( i^- \) is odd, then \( f(1)f(-1) < 0 \). Therefore, exactly one of the numbers \( f(1) \) and \( f(-1) \) is negative. But then the characteristic polynomial has at least one real root that lies either to the left of \(-1\) or to the right of \(+1\). The number of other roots in these intervals is always even and finite (counting multiplicities).

If \( i^- \) (as well as \( n \)) is even, then the characteristic polynomial may have no real roots in the intervals \((-\infty,-1)\) and \((1,\infty)\). In any case, their number in these intervals is even, which completes the proof of the congruence \((4.4)\).

Now let \( n \) be odd. Then \( f(g) \to \mp \infty \) as \( g \to \pm \infty \). If \( i^- \) is odd, then \( |B| \) the product \( f(1)f(-1) \) is positive. But then the characteristic polynomial \( f(g) \) necessarily has a real root either for \( g < -1 \) or for \( g > 1 \). The number of remaining roots in these intervals (counting multiplicities) is always even. But if \( i^- \) is even, then \( f(1)f(-1) < 0 \). Therefore, there necessarily exists a root in the interval \((-1,1)\). The number of roots to the left of \(-1\) and to the right of \(+1\) is obviously even. Therefore, the congruence \((4.4)\) is also valid in this case. The proof of the theorem is complete.

Obviously, Theorem 5 remains valid in the conservative case \((S = 0)\). Under the nondegeneracy assumptions, \( n \) is necessarily even. Indeed, by \((4.5)\), the determinant of the skew-symmetric matrix \( \Gamma \) is nonzero. But this can be the case only for even \( n \).

Let us show that \( \Omega \) is a symplectic (or canonical) mapping in this case. Indeed, consider the skew-symmetric nondegenerate bilinear form \((\Gamma x, y), x, y \in \mathbb{R}^n\). Let us prove that it is invariant under the transformation \((4.1)\). Since

\[
(\Gamma \Omega x, \Omega y) = (\Omega^T \Gamma \Omega x, y),
\]

it suffices to prove the relation \( \Omega^T \Gamma \Omega = \Gamma \). Since \( \Gamma = \Omega^T B - B\Omega \), it follows from \((4.3)\) that

\[
\Omega^T \Gamma \Omega = \Omega^T \Omega^T B \Omega - \Omega^T B \Omega \Omega = \Omega^T B - B\Omega = \Gamma.
\]

In particular, the characteristic polynomial of the operator \( \Omega \) is reciprocal by the well-known Poincaré–Lyapunov theorem. Moreover, this result can be proved directly. First, note that the relation \( \Omega^T B \Omega = B \) and the condition of nondegeneracy of the matrix \( B \) imply that \( |\Omega| = \pm 1 \). In particular, in the conservative case, the mapping \((4.1)\) preserves the standard volume in \( \mathbb{R}^n \) (with regard of a possible change of the orientation). If \( \Omega \) is the monodromy operator of the
periodic linear system (3.1), then it is homotopic to the identity mapping; consequently, $|\Omega| = 1$. Further, by (4.3) (with $S = 0$), we obtain the relations

$$
f(\varrho) = |\Omega - \varrho E| = |\Omega^T - \varrho E| = |B\Omega^{-1}B^{-1} - \varrho BB^{-1}| = |\Omega^{-1} - \varrho E|
$$

and

$$
|\Omega^{-1}| |E - \varrho \Omega| = \varrho^{-n} \left| \frac{1}{\varrho} E - \Omega \right| = (-\varrho)^n f\left(\frac{1}{\varrho}\right).
$$

Consequently, the characteristic polynomial is reciprocal for even $n$. But if $n$ is odd, then $\varrho = 1$ is a multiplier. This contradicts the assumption of nondegeneracy. In particular, for odd $n$, Theorem 5 holds only in the case of dissipation.

As an illustrative example, consider the problem on the multipliers of two-link billiard trajectories in the ellipse

$$
\frac{x^2}{a} + \frac{y^2}{b} = 1; \quad a, b > 0.
$$

To be definite, we choose a periodic motion on the segment $x = 0$, $|y| \leq \sqrt{b}$. The velocity of motion is preserved; without loss of generality, one can set $\dot{x}^2 + \dot{y}^2 = 1$. Let $\dot{x} = \sin \xi$ and $\dot{y} = \cos \xi$.

The problem on multipliers can be reduced to the analysis of the Poincaré mapping linearized in a neighborhood of the point $x = 0$, $\xi = 0$.

The elliptic billiard treated as a dynamical system admits the integral

$$
\frac{\dot{x}^2}{a} + \frac{\dot{y}^2}{b} - \frac{(\dot{x}y - x\dot{y})^2}{ab}, \quad (4.8)
$$

which is independent of the energy integral [8]. By replacing $\sin \xi$ and $\cos \xi$ by $\xi$ and $1 - \xi^2/2$, respectively, we indicate the quadratic form of the expansion of the integral (4.8) in the Maclaurin series in $x$ and $\xi$:

$$
-\frac{\xi^2}{a} + \frac{2}{a\sqrt{b}} x \xi - \frac{x^2}{ab}, \quad (4.9)
$$

If $a > b$, then this form is negative definite. Consequently, the multipliers of a periodic motion along the minor axis lie on the unit circle. Note that all of them are nondegenerate except for the case in which $a = 2b$. Then $\varrho_1 = \varrho_2 = -1$ (see [9]). But if $a < b$, then the inertia index (4.9) is equal to unity. One can readily prove the nondegeneracy of the periodic motion along the major axis of the ellipse. Consequently, by Theorem 5, its multipliers are real and different from $\pm 1$.

5. THE CAYLEY TRANSFORMATION

Let us indicate one more way of proving Theorem 5, which permits one to prove a discrete analog of Theorem 2 as well.

To a nondegenerate operator $\Omega$ (without the eigenvalues $\varrho = \pm 1$), we assign an operator $A$ by the relation

$$
E - A = 2(E - \Omega)^{-1}.
$$

It admits the inversion

$$
E - \Omega = 2(E - A)^{-1}.
$$

The operators $A$ and $\Omega$ are related by the formulas

$$
A = (E + \Omega)(E - \Omega)^{-1} = (E - \Omega)^{-1}(E + \Omega), \quad (5.1)
$$

and

$$
\Omega = (E + A)(E - A)^{-1} = (E - A)^{-1}(E + A). \quad (5.2)
$$

Each of them is referred to as the Cayley transformation. It plays an important role in the theory of matrix groups [10]. The Cayley transformation permits one to reduce the investigation of periodic linear systems to the autonomous case considered in Section 2.
Lemma 1. The substitutions (5.1) and (5.2) reduce the quadratic relation (4.3) to the linear relation
\[ BA + A^*B = \Lambda, \] (5.3)
where
\[ \Lambda = 2(E - \Omega^*)^{-1}S(E - \Omega)^{-1} \]
is a nonpositive symmetric operator.

For \( S = 0 \), this fact is well known (e.g., see [10, Chap. II]). Therefore, the quadratic form (4.2) is nonincreasing along the solutions of the linear system
\[ \dot{x} = Ax. \] (5.4)

Proof of Lemma 1. Consider the relation
\[ E + \Omega^* = A^*(E - \Omega^*) \]
adjoint to (5.1) and multiply it by \( B\Omega \) on the right. By using (4.3), we obtain
\[ B(E + \Omega) + S = A^*(B\Omega - \Omega^*B\Omega) = A^*(B\Omega - B - S) = -A^*[B(E - \Omega) + S]. \]

By multiplying it by \((E - \Omega)^{-1}\) on the right and by using (5.1), we obtain the desired relation (5.3).

The following assertion solves the problem on the relationship between the spectra of the operators occurring in the Cayley transformation.

Lemma 2. The numbers \( \varrho_1, \ldots, \varrho_n \) are the eigenvalues of the operator \( \Omega \) if and only if
\[ \lambda_j = \frac{\varrho_j - 1}{\varrho_j + 1}, \quad 1 \leq j \leq n, \]
are the eigenvalues of the operator \( A \).

Indeed, let \(|\Omega - \varrho E| = 0\). By (5.2), we have
\[ 0 = |\Omega - \varrho E| = |(E + A)(E - A)^{-1} - \varrho E| = |(E - A)^{-1}|(|\varrho + 1)A - (\varrho - 1)E| \]
\[ = |(E - A)^{-1}|(\varrho + 1)\lambda|A - \lambda E|, \]
where
\[ \lambda = \frac{\varrho - 1}{\varrho + 1}. \] (5.5)

Since \( \varrho \neq -1 \), it follows that the number (5.5) is an eigenvalue of the operator \( A \), which completes the proof.

In particular, if \( \varrho = 1 \), then \( A \) is a degenerate operator, \( \det A = 0 \). It follows from (5.5) that \( \lambda \neq 1 \). However, this property of the operator \( A \) does not play an essential role in forthcoming considerations. Note that, obviously, Lemma 2 remains valid for multiple eigenvalues.

Corollary 3. The instability degrees of the linear system (5.4) and the mapping (4.1) coincide.

Indeed, let \( \varrho = u + iv; u, v \in \mathbb{R} \). Then, by (5.5),
\[ \text{Re} \lambda = \frac{u^2 + v^2 - 1}{(u + 1)^2 + v^2}. \]

Consequently, \( \text{Re} \lambda > 0 \) if and only if \( u^2 + v^2 > 1 \) (i.e., if \(|\varrho| > 1\)).

After these preliminary remarks, Theorem 5 can be derived from the congruence (1.4) proved in [4] for linear dissipative systems of differential equations.
Theorem 6. One has the inequality \( u \leq i^- \); moreover, if \( S \) is a negative definite operator, then \( u = i^- \).

This assertion is a consequence of Theorem 2 and the remark on the coincidence of the instability degree of the linear system (5.4) and the mapping (4.1).

By way of example, consider the case in which \( i^- = 1 \). The quadratic form (4.2) acquires this form in the typical case after the loss of the property of minimum. It follows from Theorems 5 and 6 that, in this case, the operator \( \Omega \) has the only (and real) eigenvalue outside the unit disk on the complex plane, and its remaining eigenvalues lie inside the unit disk or on its boundary. In the conservative case, one can claim more: if \( i^- = 1 \), then there are two real eigenvalues \( \varrho \) and \( \varrho^{-1} \) \( (\varrho \neq \pm 1) \), and the remaining points of the spectrum lie on the unit circle.

REFERENCES