Comments on the Paper
by A. V. Borisov, A. A. Kilin, I. S. Mamaev
“How to Control the Chaplygin Ball Using Rotors. II”
Tatiana B. Ivanova* and Elena N. Pivovarova**
Udmurt State University,
ul. Universitetskaya 1, Izhevsk, 426034 Russia
Received November 16, 2013; accepted December 19, 2013

Abstract—In this paper we consider the control of a dynamically asymmetric balanced ball on
a plane in the case of slipping at the contact point. Necessary conditions under which a control
is possible are obtained. Specific algorithms of control along a given trajectory are constructed.

MSC2010 numbers: 70F40, 70E18
DOI: 10.1134/S1560354714010092
Keywords: control, dry friction, Chaplygin’s ball, spherical robot

1. INTRODUCTION
The design and development of new vehicles having a spherical form have recently given impetus
to active research on the dynamics and possibility of controlling spherical robots with various
internal mechanisms [1–4, 7–10]. One of the examples is a mechanism in which variable gyroscopic
momentum is generated by rotors [2, 3, 5, 6, 9]. The authors of [2, 9] investigate the model of
motion of this system based on non-holonomic no-slip constraints (i.e., the velocity of the contact
point is zero).
On the other hand, slipping is present, as a rule, in real systems and has a significant influence
on the dynamics of the entire system. The problem of control by specifying the rotational velocities
of the rotors with viscous and dry friction was considered in [3]. However, in this paper there was an
inaccuracy in the solution of the system of equations for projections of the velocity of the contact
point in the case of dry friction. As a result, it was concluded that the control of the system along
the prescribed trajectory is impossible. In this note we show that the control is possible (albeit
with restrictions) and construct specific algorithms for the control along various trajectories.

2. CONTROL WITH DRY FRICTION
Consider the rolling of a dynamically asymmetric balanced ball (the center of mass coincides
with the geometric center) on a plane with slipping at the contact point. The equations of motion
are
\[ m\ddot{V} = F, \quad (\tilde{I}\Omega + K)\cdot = R \times F, \]
where \( m \) is the mass of the ball, \( V \) and \( \Omega \) are the velocity of the ball’s center and its angular
velocity, \( K \) is the gyroscopic momentum of the rotors, \( F \) is the friction force acting at the point of
contact of the ball with the plane, \( R = -Re_z \) is the vector directed from the center of mass to the
point of contact, and \( \tilde{I} \) is the tensor of inertia of the ball relative to the center of mass in the fixed
axes. This tensor is related to the mass moment of inertia tensor \( I = \text{diag}(I_1, I_2, I_3) \) by
\[ \tilde{I} = Q^T I Q, \]
where \( Q \) is the orthogonal matrix parameterizing the rotation of the moving axes \( Ce_1e_2e_3 \) relative to the fixed axes \( Oe_xe_ye_z \) (see Fig. 1). In order for the velocity of the center to remain parallel to the plane, i.e., \( V_z \equiv 0 \), we shall assume that the force in Eqs. (1) also satisfies the condition \( F_z \equiv 0 \).

These equations must be supplemented with kinematic relations governing the rotation of the moving axes and the motion of the center of mass. They can be represented in matrix and scalar form as

\[
\dot{Q} = Q\tilde{\Omega}, \quad \tilde{\Omega}_{ij} = \varepsilon_{ijk}\Omega_k, \\
\dot{x} = V_x, \quad \dot{y} = V_y,
\]

where \( \varepsilon_{ijk} \) is the Levi-Civita tensor and the indices \( i, j, k \) take the values \( x, y, z \).

The system of equations (1) and (3) admits a vector integral, which is the angular momentum relative to the point of contact:

\[
M = \tilde{I}\Omega + K + mV \times R = \text{const}.
\]

In the case of dry friction

\[
F = -\mu mg\frac{V_p}{|V_p|} = -\mu mg\frac{V + \Omega \times R}{|V + \Omega \times R|},
\]

where \( V_p \) is the velocity of the contact point, \( \mu \) is the coefficient of dry friction, and \( g \) is the free-fall acceleration.

Our goal is to determine the control gyroscopic momentum \( K \) during the motion of the ball’s center in a prescribed trajectory \( r(s(t)) \), where \( s(t) \) is the law of motion in a trajectory.

It follows from (1) and (5) that

1) for the acceleration of the ball’s center of mass the following relation holds:

\[
\ddot{r}^2 = \mu^2 g^2;
\]

2) we can determine only the direction of the velocity of the contact point but not its absolute value.

Indeed, the velocity of the contact point can be represented as

\[
V_p = -|V_p|\frac{\dot{r}}{\mu g},
\]

and since \( |\dot{r}| = \mu g \), \( V_p = -|V_p|i \), where \( i = \frac{\dot{r}}{|\dot{r}|} \) is the unit vector directed along the vector of acceleration \( \dot{r} \). Thus, \( |V_p| = \lambda \) can be any positive definite function. Generally, during the motion in a given trajectory, \( \lambda(t) \) is the function of time. If the control involves maintaining a constant velocity of the contact point, we have to choose \( \lambda = \text{const} \).
**Proposition.** For any smooth trajectory \( r(s) \) there exists a control where the ball’s center will move in a prescribed trajectory according to the law \( s(t) \) which satisfies the equation
\[
\ddot{s}^2 + k(s)^2 \dot{s}^4 = \mu^2 g^2,
\] (7)
where \( k(s) \) is the curvature of the trajectory. Note that there is an arbitrariness in the choice of the function \( \Omega_z(t) \), which can be used for some additional orientation of the ball either in the process of motion or at the endpoint of the trajectory.

**Proof.** Since the angular momentum relative to the point of contact (4) remains constant, on the zero level set of this integral the gyroscopic momentum of rotors defining the control can be obtained from the following relation
\[
K = -\frac{1}{R} \mathbf{I} \left( \left( \frac{\hat{r}}{\mu g} \lambda - \hat{r} \right) \times k \right) - m\dot{r} \times R,
\] (8)
where \( k = (0, 0, 1) \).
Thus, to find the control, it is necessary
- to determine \( \Omega_x(t) \) and \( \Omega_y(t) \) from the first of Eqs. (1) using the expression for friction force (5);
- to find the control \( K(t) \) using (8).

\[\Box\]

**Remark.** In this case the trajectory of motion can be arbitrary but the law of motion is strictly fixed by Eq. (7).

**Example 1.** Consider the motion of the ball in a straight line along the axis \( Ox \) with the constant acceleration \( a_0 = \mu g \) according to the law
\[
x(t) = \frac{\mu g t^2}{2}, \quad y(t) = 0.
\]
We find the angular velocity of the ball from the first of Eqs. (1) with a given value \( \lambda = \text{const} \)
\[
\Omega_x(t) = 0, \quad \Omega_y(t) = \frac{\dot{x}}{R} + \frac{\lambda \ddot{x}}{\mu g R} = \frac{\mu g}{R} t + \frac{\lambda}{R}, \quad \Omega_z(t) = 0.
\]
Then the gyroscopic momentum is
\[
K_1 = K_3 = 0, \quad K_2 = -\frac{I_2}{R} \lambda - \mu g \left( \frac{I_2}{R} + mR \right) t.
\]
Thus, since the angular velocity is not zero at the initial instant of time, the parameter \( \lambda \) defines the initial slipping at the contact point.

**Example 2.** We consider the motion of the ball in an arc of a circle of radius \( r \) according to the law
\[
x(t) = r \cos \omega t, \quad y(t) = r \sin \omega t,
\]
and choose the value \( \omega \) so that the condition (6) holds, i.e., \( \omega = \sqrt{\frac{\mu g}{r}} \). In this case the angular velocities are
\[
\Omega_x(t) = -\frac{1}{R} \left( \frac{\dot{y} + \lambda \ddot{y}}{\mu g} \right) = \frac{r \omega}{R} \left( \cos \omega t - \frac{\lambda \omega}{\mu g} \sin \omega t \right),
\]
\[
\Omega_y(t) = \frac{1}{R} \left( \frac{\dot{x} + \lambda \ddot{x}}{\mu g} \right) = \frac{r \omega}{R} \left( \sin \omega t + \frac{\lambda \omega}{\mu g} \cos \omega t \right).
\]
Substituting the relations obtained into (8), we find the dependence for the gyroscopic momentum shown in Fig. 2.
Fig. 2. The control gyroscopic momentum of the rotors during the motion of the ball in an arc of a circle for the time during which the ball passes a full circle with the parameters: $\mu = 0.2$, $I = \text{diag}(2, 3, 4)$, $R = 1$, $r = 0.5$. The left figure shows the dependence for $\lambda = 1$ and the right figure for $\lambda = 4$.

ACKNOWLEDGMENTS

The authors thank A. V. Borisov, I. S. Mamaev and A. A. Kilin for useful discussions and remarks.

This work was supported by Analytic Departmental Target Program “Development of Scientific Potential of Higher Schools” for 2012–2014, no. 1.1248.2011, the Grant of the President of the Russian Federation for Support of Leading Scientific Schools NSh-2964.2014.1, the grant of the President of the Russian Federation for the Support of Young Doctors of Science (MD-2324.2013.1) and Candidates of Science (MK-2171.2014.1).

REFERENCES